Digital Signal Processing

POSTECH

Department of Electrical Engineering

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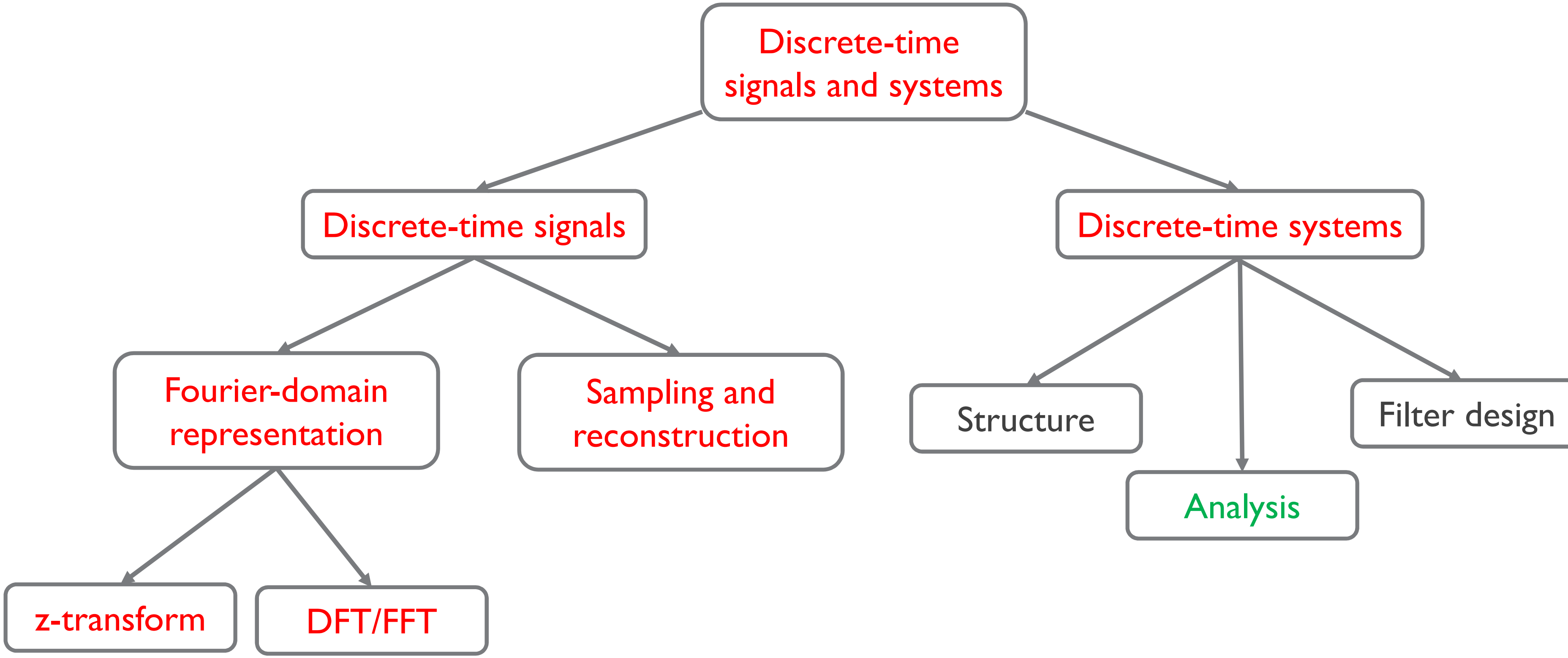
Course at glance



FIR vs. IIR Filters

Impulse response for rational system function

- Consider the system with only 1st-order poles (assuming $M>N$)

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Assuming the system to be causal

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

IIR vs. FIR systems

- $H(z)$ may have (multiple) poles only at z=0 due to pole/zero cancellations

- If there is at least one nonzero pole of $H(z)$ not cancelled by a zero
  ➔ The impulse response $h[n]$ will have at least one $A_k(d_k)^n u[n]$
  ➔ IIR system

- If $H(z)$ has no poles except at z=0

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}, \;\; h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

  ➔ FIR system

Simple FIR example

- Consider FIR system

$$h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

z=a zero cancels pole

- System function

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}$$

- Input-output relation

$$y[n] = \sum_{k=0}^{M} a^k x[n-k] \qquad y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$

➔ Two expressions represent identical systems

Minimum-Phases Systems

Minimum-phases systems

- To have causal and stable systems
  → Poles must be inside the unit circle but no restriction on zeros

- To have causal and stable inverse
  → Zeros must be inside the unit circle as well

- If all poles and zeros are inside the unit circle
  → Such systems are referred to as minimum-phases systems

Minimum-phase and all-pass decomposition

- Any causal, stable rational function can be decomposed as

$$H(z) = H_{\min}(z)H_{\mathrm{ap}(z)}$$

Minimum-phase system　　All-pass system

$|c| < 1$

- Proof: suppose $H(z)$ has one zero outside the unit circle at $z = 1/c^*$ and remaining poles and zeros are inside the unit circle

$$H(z) = H_1(z)(z^{-1} - c^*) = \underbrace{H_1(z)(1 - cz^{-1})}_{\text{Minimum-phase system}} \frac{z^{-1} - c^*}{(1 - cz^{-1})}$$

Minimum-phase system

All-pass system

- Possible to generalize to multiple zeros outside the unit circle

Important property

- Let $H(z) = H_{\min}(z)H_{\mathrm{ap}(z)}$

- Frequency-response relationship

$$|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|$$

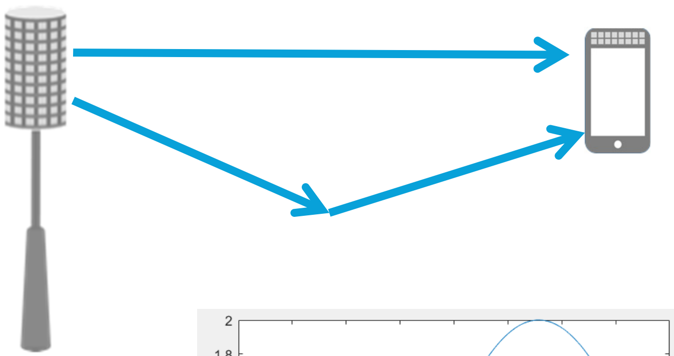
for all ω because

$$|H_{\mathrm{ap}}(e^{j\omega})| = 1$$

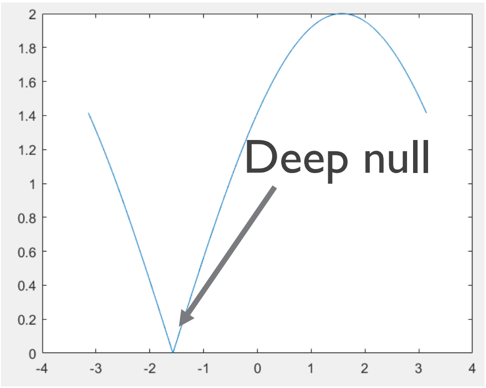
for all ω

Frequency-response compensation

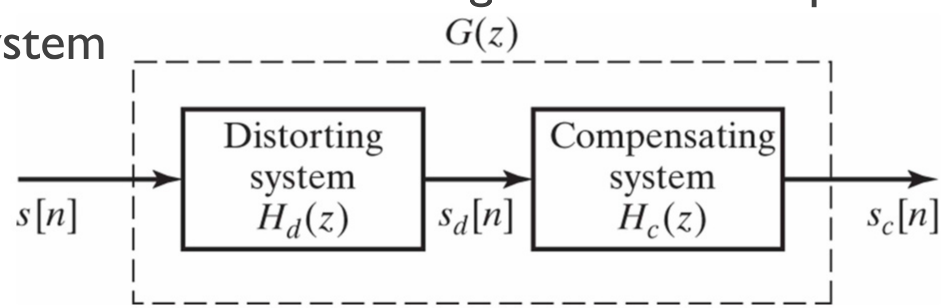
- Signals can be distorted by an LTI system with an undesirable frequency response
  - Example: two-path communication channel



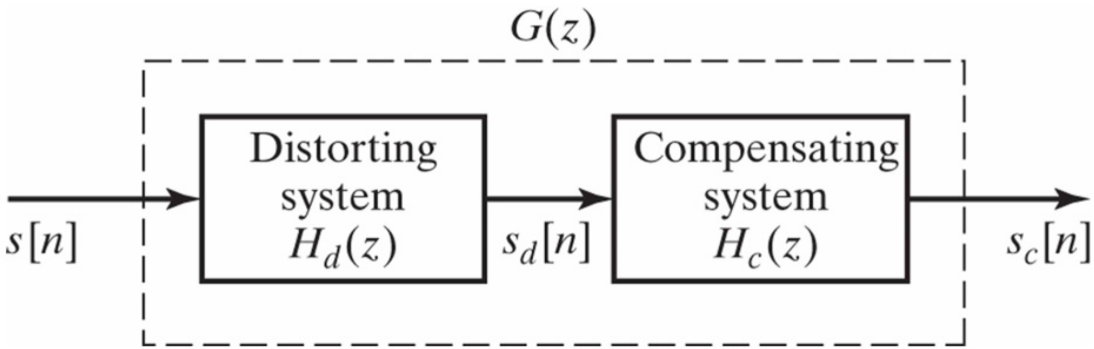
$$q(t) = \underbrace{\delta(t) - e^{j\phi}\delta(t - T_0)}_{\text{Two-way multipath}}$$



- Process the distorted signal with a compensating system



Perfect compensation



- With perfect compensation $s_c[n] = s[n]$
  ➔ $H_c(z)$ is the inverse of $H_d(z)$

- We are interested in stable and causal distorting and compensating systems
  ➔ Perfect compensation is possible only if $H_d(z)$ is a minimum-phase system

- Not all distorting systems are minimum-phase systems

Non-ideal compensation systems

- Decompose $H_d(z) = H_{d\,\min}(z) H_{\text{ap}}(z)$

  Minimum-phase system   All-pass system

- Choose the compensating filter

$$H_c(z) = \frac{1}{H_{d\,\min}(z)}$$

- Overall system becomes

$$G(z) = H_d(z) H_c(z) = H_{\text{ap}}(z)$$

  - Frequency-response magnitude exactly compensated
  - Phase response modified to $\angle H_{\text{ap}}(e^{j\omega})$

Relation b/w magnitude and phase

- $|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)\,|_{z=e^{j\omega}}$

- Consider linear constant coefficient difference equation

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}, \qquad H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}$$

$$H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

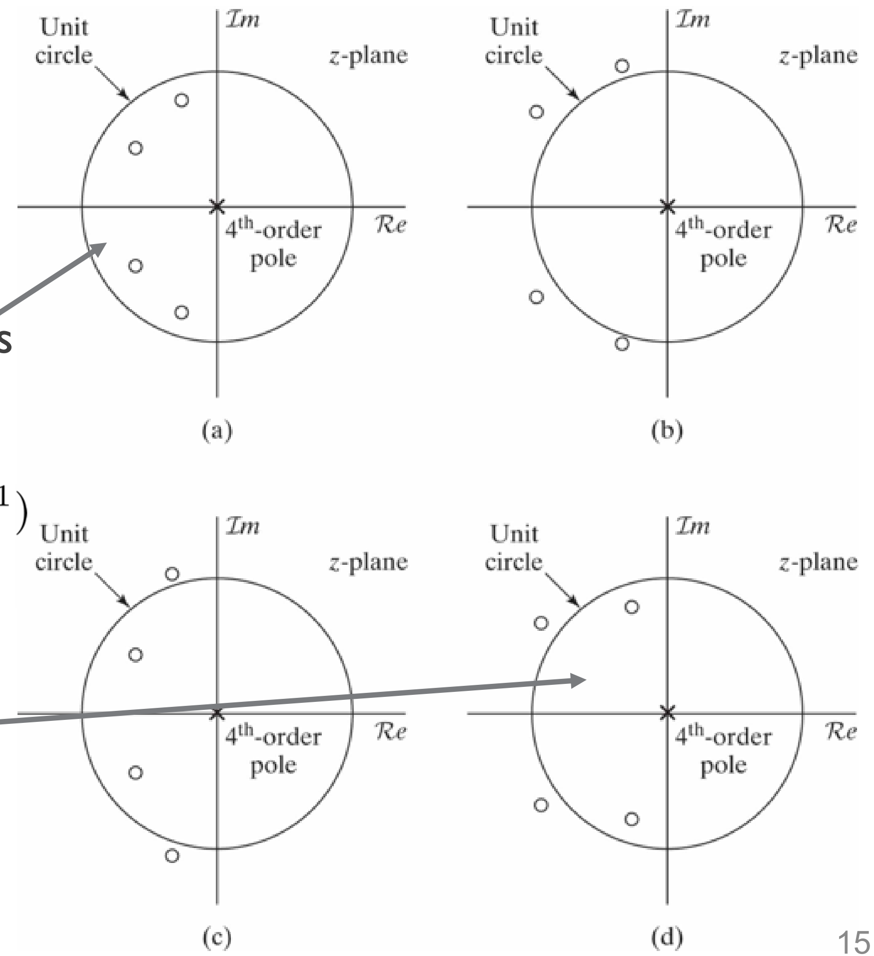
- For same magnitude response, both c_k and $1/c_k^*$ are possible zeros
  - What about poles???

Properties of minimum-phase system I

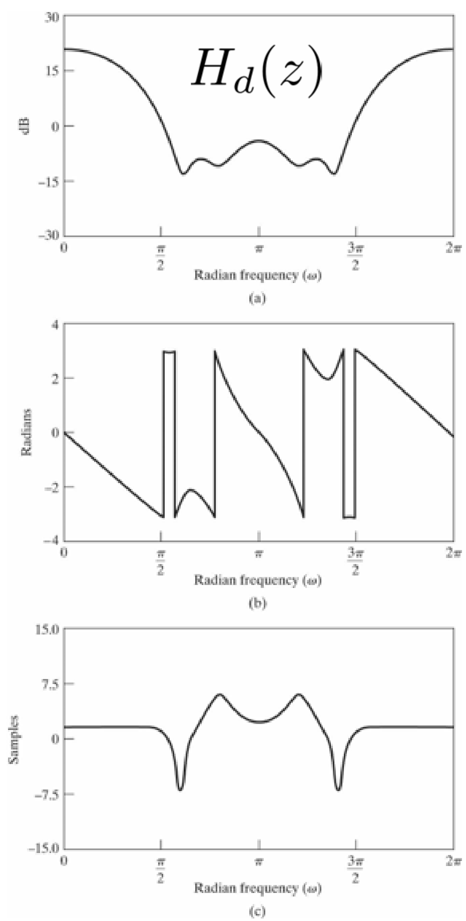
- For a system with *M* pairs of zeros
  - 2^M possible causal & stable systems with the same frequency-response magnitude $|H(e^{j\omega})|$
  - Only one minimum-phase system exists ➔ All zeros inside unit circle

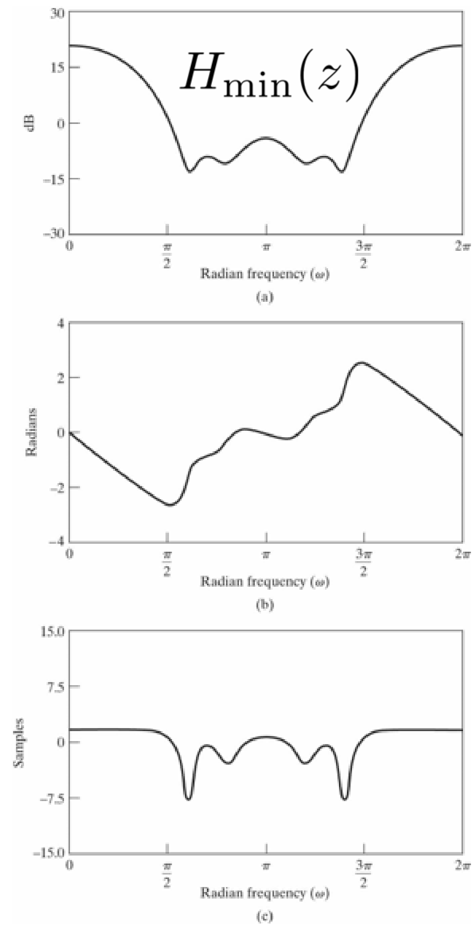
$$H_{\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})$$
$$\times\,(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

$$H_d(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})$$
$$\times\,(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$$



Frequency-response plots





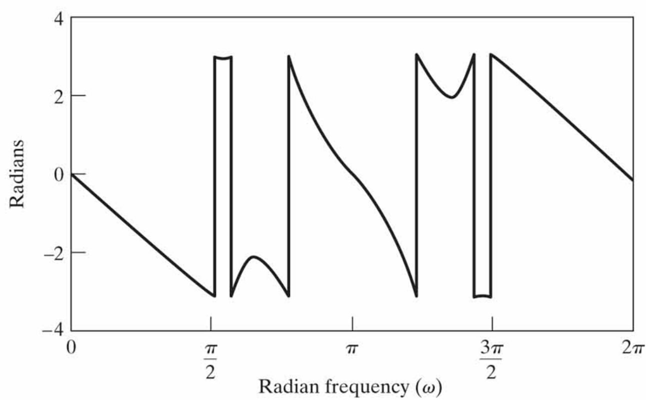
Minimum phase-lag property

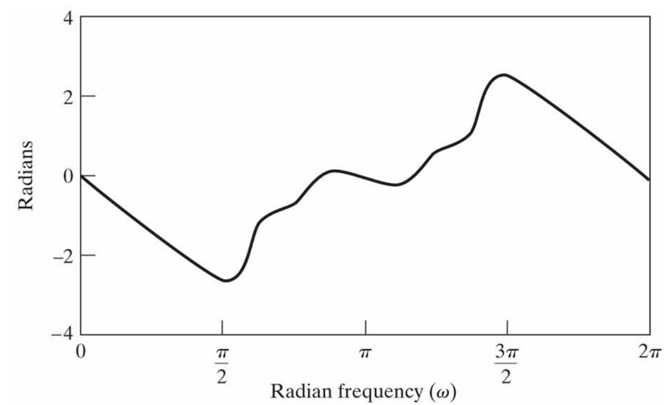
- Define the negative of the phase as "phase-lag"
  ➔ Larger the phase, smaller the phase-lag

Always negative in $0 \leq \omega \leq \pi$

- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the minimum phase-lag function for $0 \leq \omega \leq \pi$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{\mathrm{ap}}(e^{j\omega})]$$



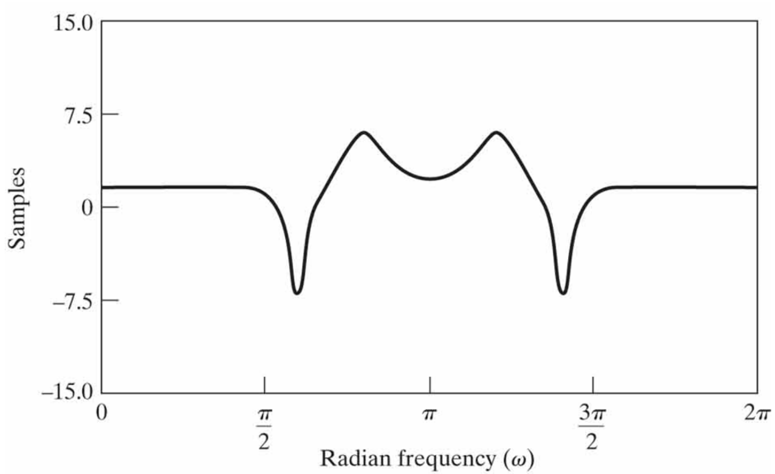


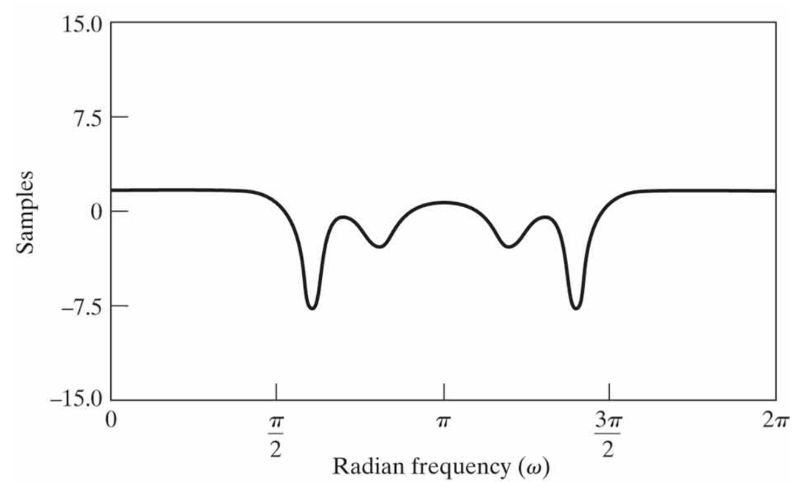
Minimum group-delay property

- Clearly $\mathrm{grd}[H(e^{j\omega})] = \mathrm{grd}[H_{\min}(e^{j\omega})] + \mathrm{grd}[H_{\mathrm{ap}}(e^{j\omega})]$

  Always positive in $0 \leq \omega \leq \pi$

- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the minimum group delay





Minimum energy-delay property

- All systems that have the same frequency-response magnitude has equal energy

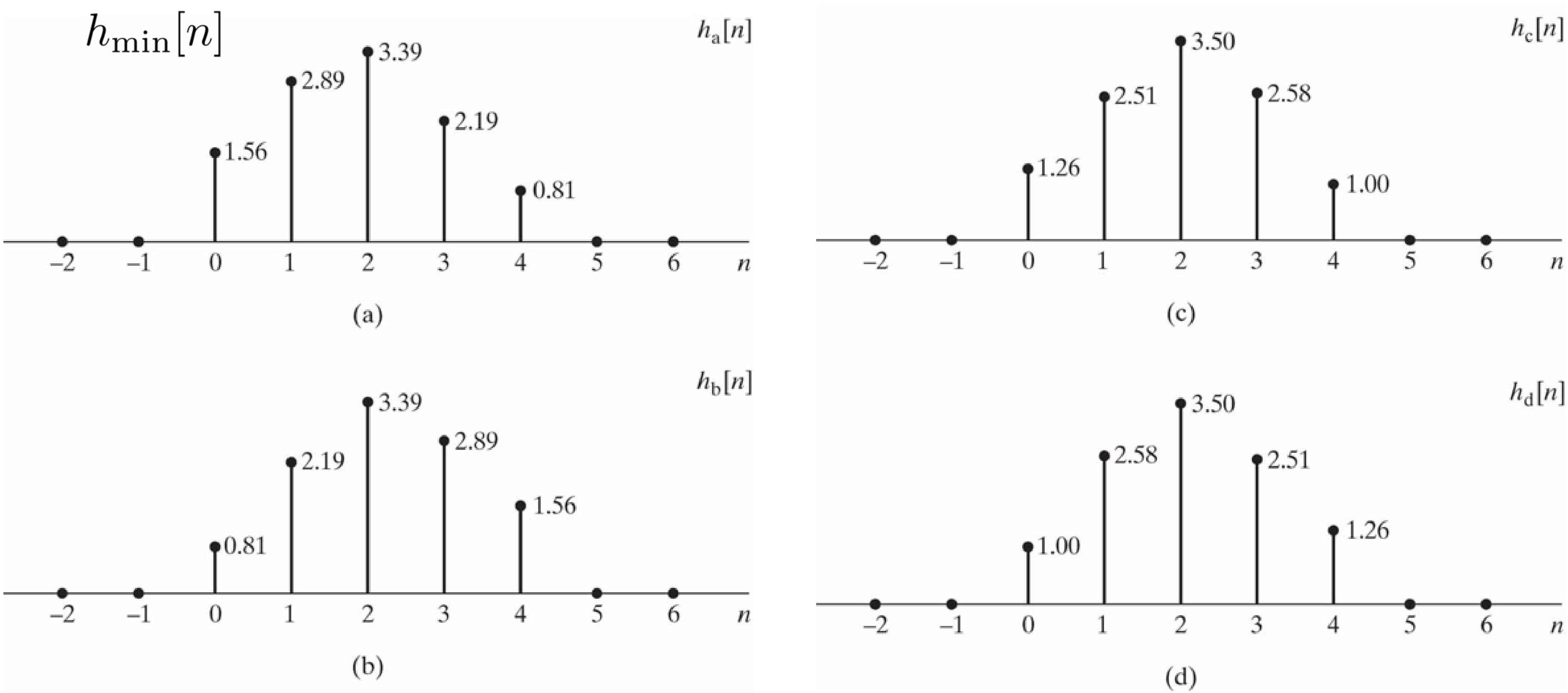
$$\sum_{n=0}^{\infty} |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{\min}(e^{j\omega})|^2 d\omega = \sum_{n=0}^{\infty} |h_{\min}[n]|^2$$

- Define partial energy $E[n] = \sum_{m=0}^{n} |h[m]|^2$

- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the most energy concentrated around n=0

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{\min}[m]|^2$$

Minimum energy-delay property



Linear phase systems

- For causal systems, zero phase is not possible
  - Some phase distortion must be allowed
- In many situations, it is desirable to design systems to have exactly or approximately linear phase
- Ideal delay system example

$$H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega\alpha}, \ \ |\omega| < \pi$$

$$|H_{\mathrm{id}}(e^{j\omega})| = 1$$

$$\angle H_{\mathrm{id}}(e^{j\omega}) = -\omega\alpha$$

$$\mathrm{grd}[H_{\mathrm{id}}(e^{j\omega})] = \alpha$$

  - α does not have to be an integer (See 5.7.1)

Generalized linear phase

- Generalized linear-phase system is defined as

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$

$$\alpha, \beta: \ \text{real constants}$$

$$A(e^{j\omega}): \ \text{a real function of } \omega$$

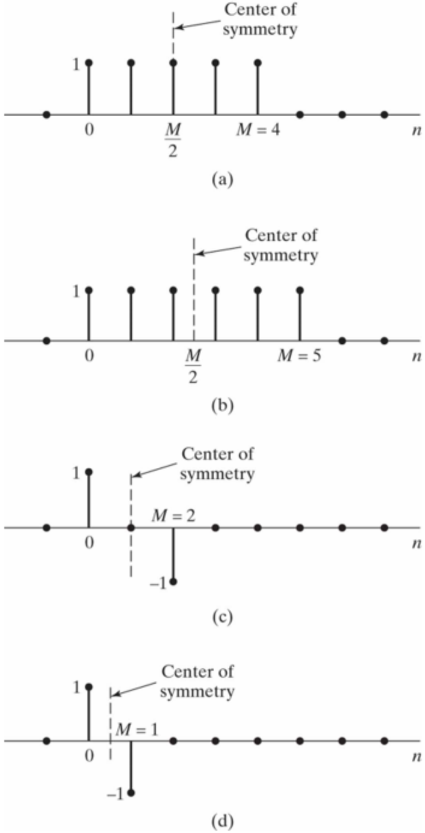
- Phase and group delay

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha$$

$$\text{grd}[H(e^{j\omega})] = \alpha$$

Causal FIR generalized linear-phase systems

- Four classes of FIR systems with generalized linear-phase
  - Type I
    - Symmetric: $h[n] = h[M-n]$
    - M even
  - Type II
    - Symmetric: $h[n] = h[M-n]$
    - M odd
  - Type III
    - Antisymmetric: $h[n] = -h[M-n]$
    - M even
  - Type IV
    - Antisymmetric: $h[n] = -h[M-n]$
    - M odd



Locations of zeros for FIR linear-phase systems

- For Types I and II, channel impulse responses are symmetric $h[n] = h[M-n]$
- System function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M} = z^{-M}H(z^{-1})$$

- If z_0 is a zero of $H(z)$, then $H(z_0) = z_0^{-M}H(z_0^{-1}) = 0$
  ➔ z_0^{-1} is also a zero
- If h[n] is real and z_0 is a zero of $H(z)$, then z_0^* is also a zero
  ➔ $(z_0^*)^{-1}$ is also a zero
- $H(z)$ will have factors of the form

$$(1 - rz^{-1})(1 - r^*z^{-1})(1 - r^{-1}z^{-1})(1 - (r^{-1})^*z^{-1})$$

  - What if zeros are on the unit circle?

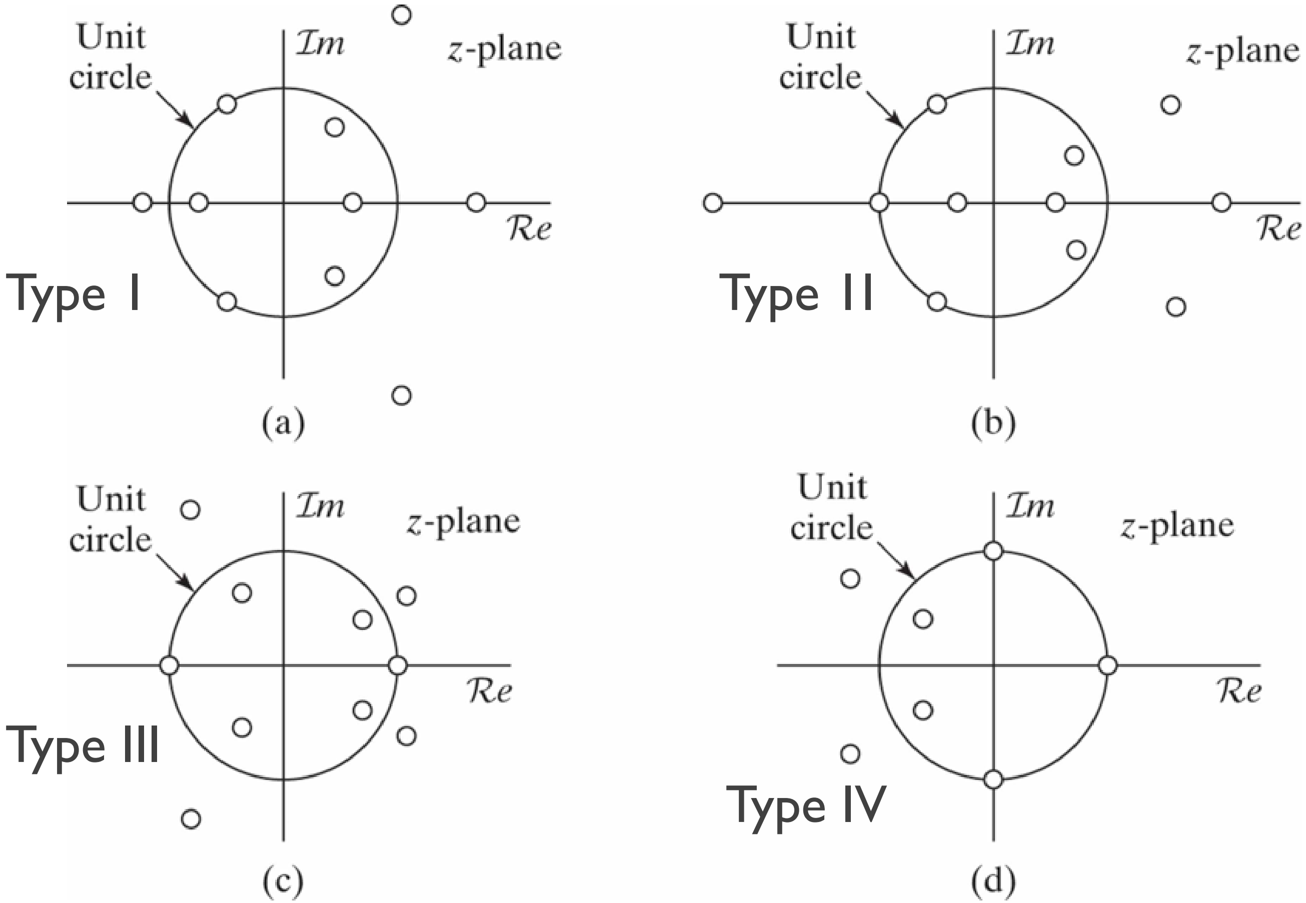
Locations of zeros for FIR linear-phase systems

- For Types III and IV, $h[n] = -h[M-n]$
- System function $H(z) = -z^{-M} H(z^{-1})$

- If z=1, $H(1) = -H(1)$ ➔ z=1 is always a zero
- If z=-1, $H(-1) = (-1)^{-M+1} H(-1)$
  ➔ If *M* is even, z=-1 should be a zero

- These constraints are important in FIR linear-phase filter designs
  - Example: with (anti)symmetric impulse response, $z = -1$ $(\omega = \pi)$ should be always zero with *M* (even)odd
    ➔ For highpass filter with (anti)symmetric impulse response, *M* should be (odd)even!

Typical plots of zeros for linear-phase systems



IIR filter and linear-phase

- So far, we discussed FIR linear-phase filters
- Can IIR filters have a linear-phase response?

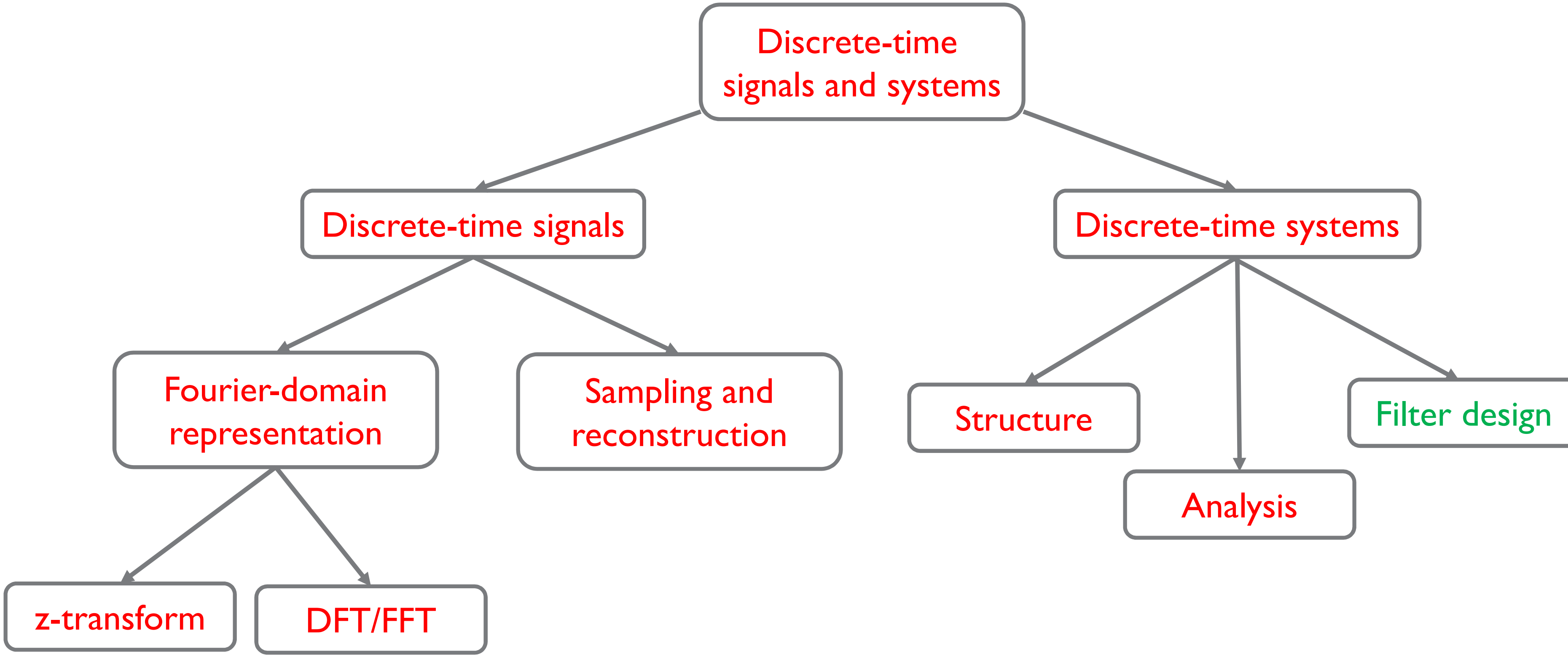
- Check with the same criterion

$$H(z) = \pm z^{-M} H(z^{-1})$$

  - If p_0 is a pole of $H(z)$, then $1/p_0$ is also a pole
  - If h[n] is real, then p_0^* and $1/p_0^*$ are also poles

  ➔ Cannot be causal and stable!!!

Course at glance

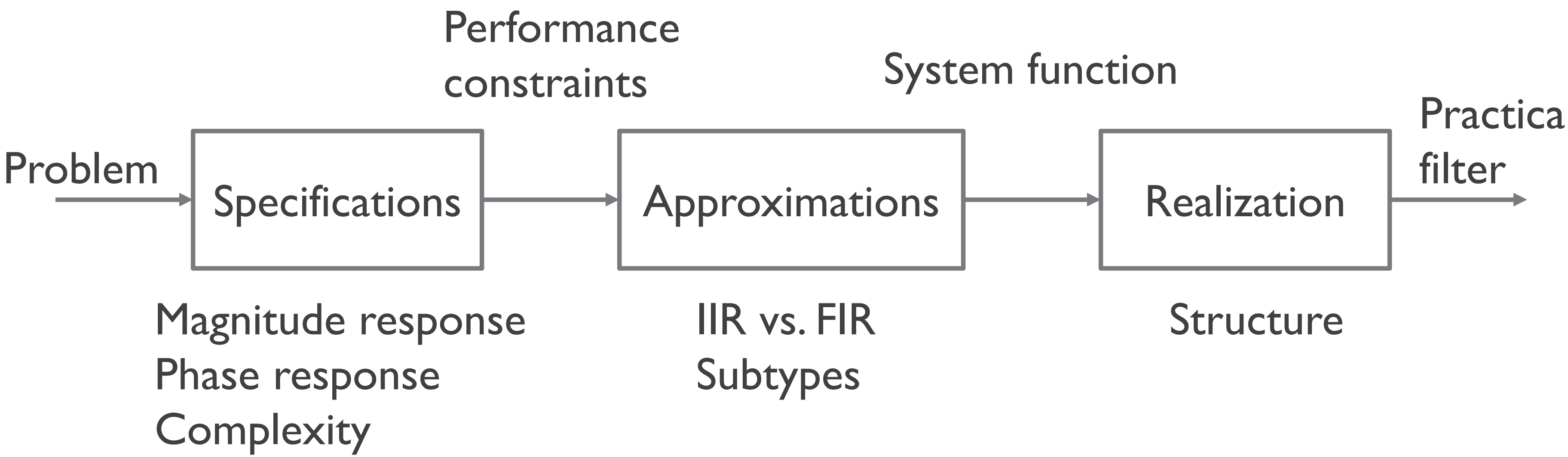


Definition of filter

- Filter, in broader sense, covers any system
  - Distortion environments are also filters
- We denote filters as controllable systems here

Filter design process

- Three design steps



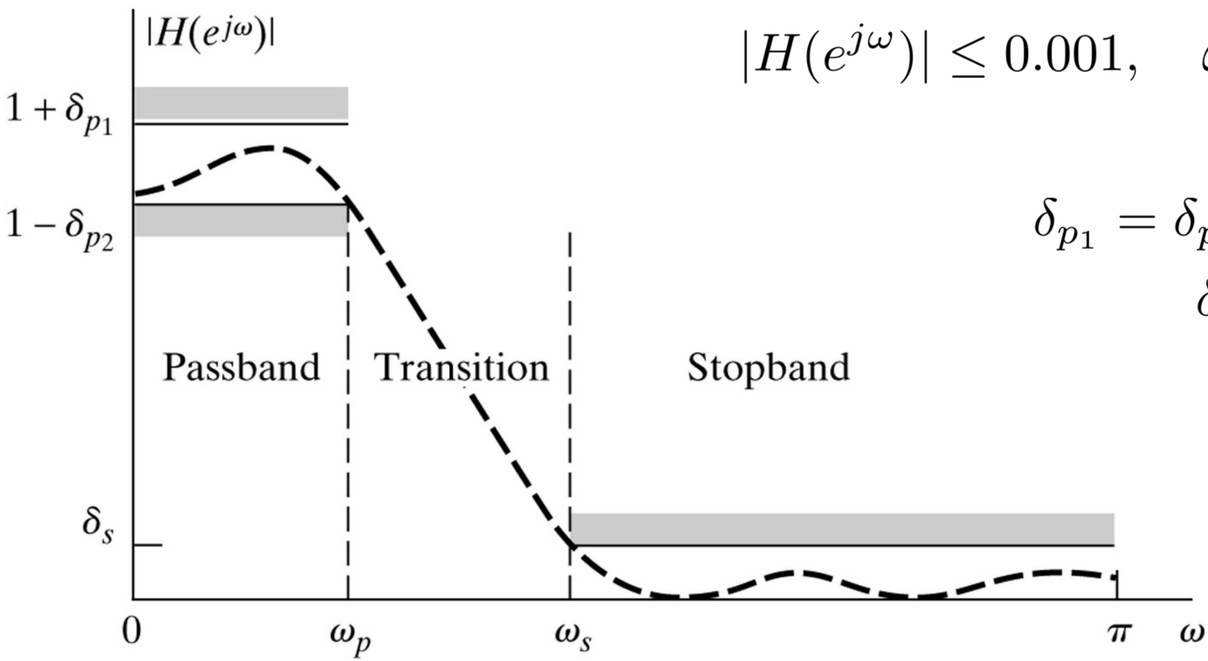
- Focus on lowpass filters
  - Can be generalized to other frequency-selective filters

Example specifications

- Specifications for a discrete-time lowpass filter

$$1 - 0.01 \le |H(e^{j\omega})| \le 1 + 0.01, \quad 0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le 0.001, \quad \omega \ge \omega_s$$



$$\delta_{p_1} = \delta_{p_2} = 0.01$$
$$\delta_s = 0.001$$

Specifications of frequency response

- Typical lowpass filter specifications in terms of tolerable
  - Passband distortion → as **smallest** as possible
  - Stopband attenuation → as **greatest** as possible
  - Width of transition band → as **narrowest** as possible
- Improving one often worsens others → tradeoff exists
- Increasing filter order may improve all → increase complexity

Design a filter

- Design goal
  ➔ Find system function to make frequency response meet the specifications (tolerances)

- Infinite impulse response (IIR) filter
  - Poles insider unit circle due to causality and stability
  - Rational function approximation

- Finite impulse response (FIR) filter
  - For filters with linear phase requirement
  - Polynomial approximation

Example of IIR filter design

- For rational (and stable and causal) system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

find the system coefficients such that the corresponding frequency response

$$H(e^{j\omega}) = H(z)\,\big|_{z=e^{j\omega}}$$

provides a good approximation to a desired response

$$H(e^{j\omega}) \approx H_{\text{desired}}(e^{j\omega})$$

IIR vs. FIR

- Either FIR or IIR is often dependent on the phase requirements

- Only FIR filter can be at the same time stable, causal and GLP

- Design principle
  - If GLP is required ➔ FIR
  - If not ➔ IIR preferable because IIR can meet specifications with lower complexity

IIR vs. FIR

- IIR
  - Rational system function
  - Poles and zeros
  - Stable/unstable
  - Hard to control phase
  - Low order (4-20)
  - Designed on the basis of analog filter
- FIR
  - Polynomial system function
  - Only zeros
  - Always stable
  - Easy to get (generalized) linear phase
  - High order (20-200)
  - Usually unrelated to analog filter designs

IIR Filter Design

Discrete-time IIR filters from continuous-time filters

- Continuous-time (or analog) IIR filter design is highly advanced
  - Relatively simple closed-form design possible

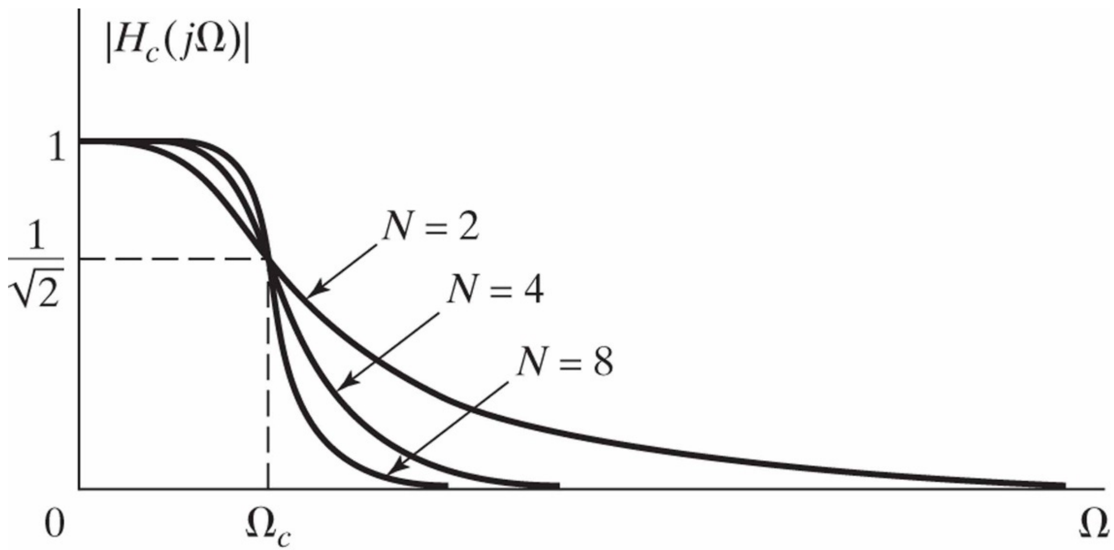
- Discrete-time IIR filter design
  - Filter specifications for discrete-time filter
  - Convert to continuous-time specifications
  - Design continuous-time filter
  - Convert to discrete-time filter
    - Impulse invariance method
    - Bilinear transformation method

Analog filter designs

- Butterworth filter
- Type I Chebyshev filter
- Type II Chebyshev filter
- Elliptic filter

Butterworth lowpass filter

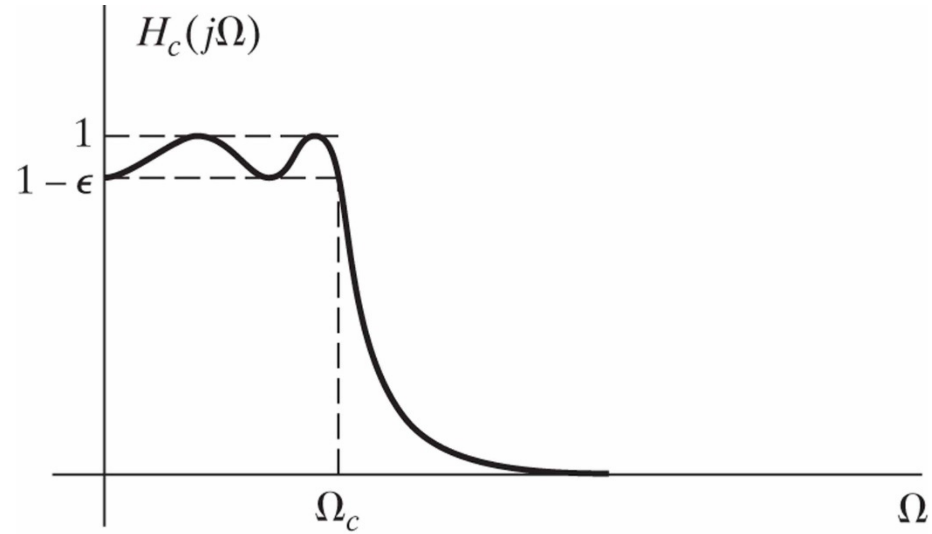
- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$
  - Two parameters
    - Order N
    - Cutoff frequency Ω_c
  - Monotonic in both passband and stopband



Type I Chebyshev lowpass filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 V_N^2(\Omega/\Omega_c)}$

  where $V_N(x) = \cos(N \cos^{-1} x)$
  - Three parameters
    - Order N
    - Cutoff frequency Ω_c
    - Allowable passband ripple ϵ



- $|H_c(j\Omega)|^2$ has equi-ripple error in passband and monotonic in stopband

Type II Chebyshev lowpass filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1 + [\epsilon^2 V_N^2(\Omega/\Omega_c)]^{-1}}$

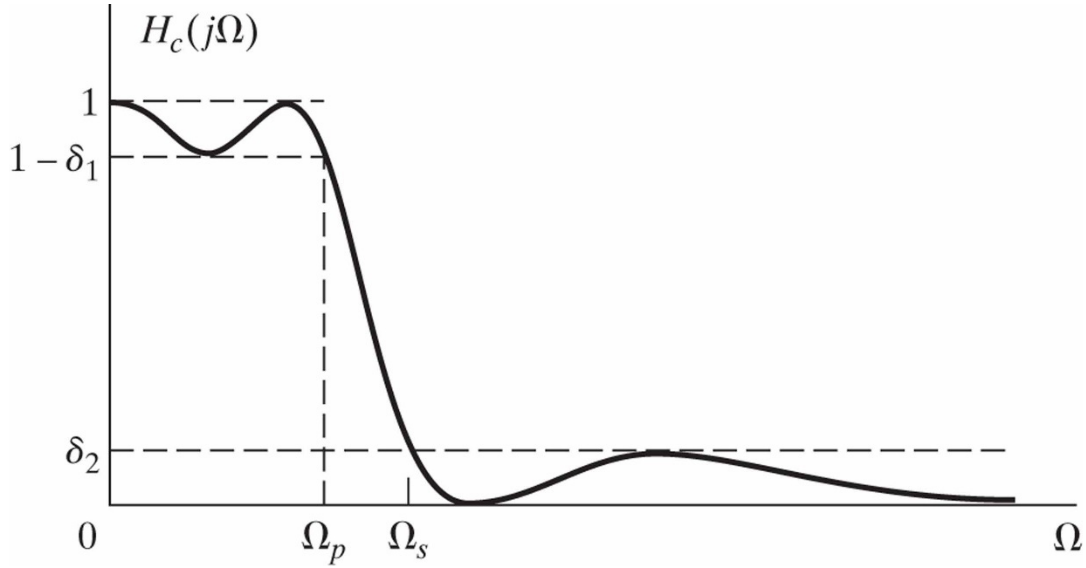
- Similar to Type I Chebyshev lowpass filter
  - $|H_c(j\Omega)|^2$ now has equi-ripple error in stopband and flat in passband

Elliptic filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 U_N^2(\Omega)}$

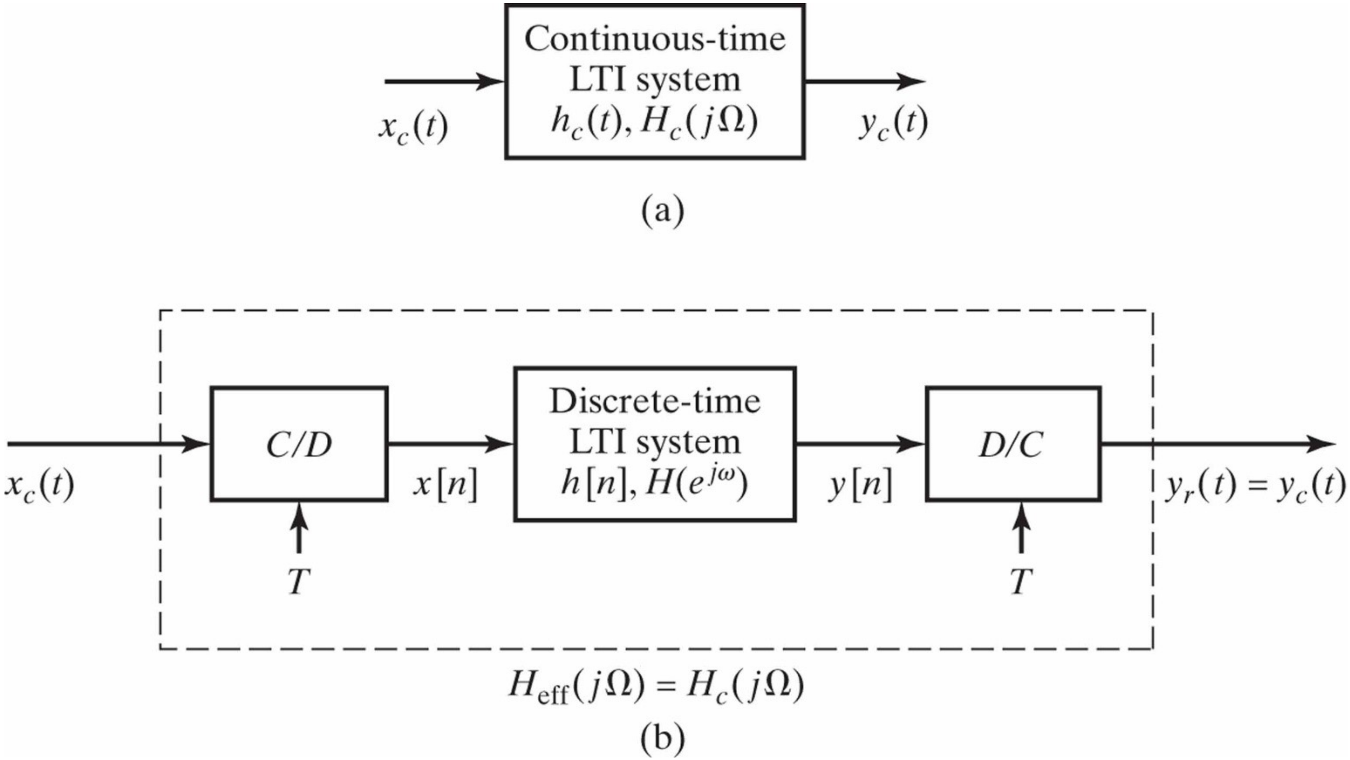
  where $U_N(\Omega)$ is a Jacobian elliptic function

- $|H_c(j\Omega)|^2$ has equi-ripples in both passband and stopband



Discrete-time IIR filter design – impulse invariance

- Recall "discrete-time processing of continuous-time signals" in Section 4.4



Output signal

- Necessary conditions
  - The discrete-time system is LTI
  - Continuous-time signal $x_c(t)$ is bandlimited
  - Sampling rate Ω_s is at or above the Nyquist rate $2\Omega_N$

- If all conditions are satisfied, the output signal becomes

$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega)$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Cutoff frequency of ideal lowpass filter

$$\omega = \Omega T$$

Impulse invariance

- Recall $H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$

- We want to have $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

$$\Rightarrow \quad H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi$$

- In time-domain: $h[n] = Th_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

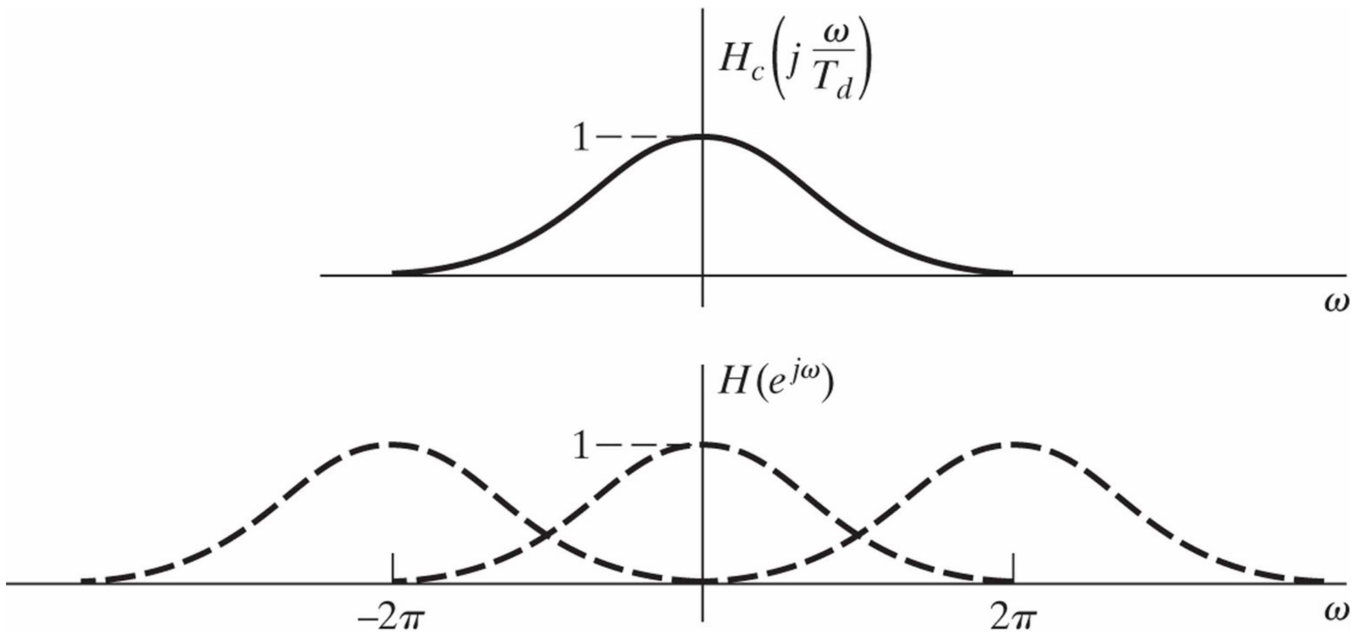
$$\begin{aligned} H(e^{j\omega}) &= T\frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \\ &= H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi \end{aligned}$$

Because $H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$

Only true when the filter is bandlimited

Impulse invariance - aliasing

- If the analog filter is not bandlimited (typically the case in practice)
  ➔ Aliasing occurs in the discrete-time filter
  - Impulse invariance not appropriate for designing highpass filters



How can we avoid the aliasing?

- Consider higher sampling frequency for analog filter $\Omega_s = 1/T$
- Will this work? No!
  - Filter specifications given from discrete-time filter requirements
  - The specifications transformed to continuous-time by $\Omega = \omega/T$
  - Continuous-time filter designed by continuous-time specifications
  - Final discrete-time filter obtained by impulse invariance method (sampling)

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

→ Effect of $\Omega_s = 1/T$ cancels out

- Aliasing can be avoided by overdesigning analog filter

Interpretation using system functions

- Transformation from continuous-time system to discrete-time system is easy to carry out using system functions
- After partial fraction expansion

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} h[n] &= T_d h_c(nT_d) \\ &= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \\ &= \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n] \end{aligned}$$

$$\xleftrightarrow{\mathcal{Z}} H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Interpretation using system functions

- Mapping from $H_c(s)$ to $H(z)$
  - Pole of $H_c(s)$ at $s = s_k$ maps to pole of $H(z)$ at $z = e^{s_k T_d}$
    ➔Stability and causality preserved
  - Continuous-time: $\mathrm{Re}\{s_k\} < 0$
  - Discrete-time: $z = e^{s_k T_d}$ inside the unit circle

Impulse invariance with Butterworth filter

- Specifications: $0.89125 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi$

$$|H(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi$$

- Since the effect $\Omega_s = 1/T$ cancels out, set *T*=1 and $\omega = \Omega$

- Transformed analog specifications

$$0.89125 \leq |H_c(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq 0.2\pi$$

$$|H_c(j\Omega)| \leq 0.17783, \quad 0.3\pi \leq |\Omega| \leq \pi$$

- Due to monotonicity of Butterworth filter

$$|H_c(j0.2\pi)| \geq 0.89125$$

$$|H_c(j0.3\pi)| \leq 017783$$

Impulse invariance with Butterworth filter

- The magnitude-squared function of Butterworth filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- From the specifications $|H_c(j0.2\pi)| \geq 0.89125, \quad |H_c(j0.3\pi)| \leq 017783$

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2, \quad 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

  - Simultaneous solutions are $N = 5.8858, \;\; \Omega_c = 0.70474$

    Should be integer

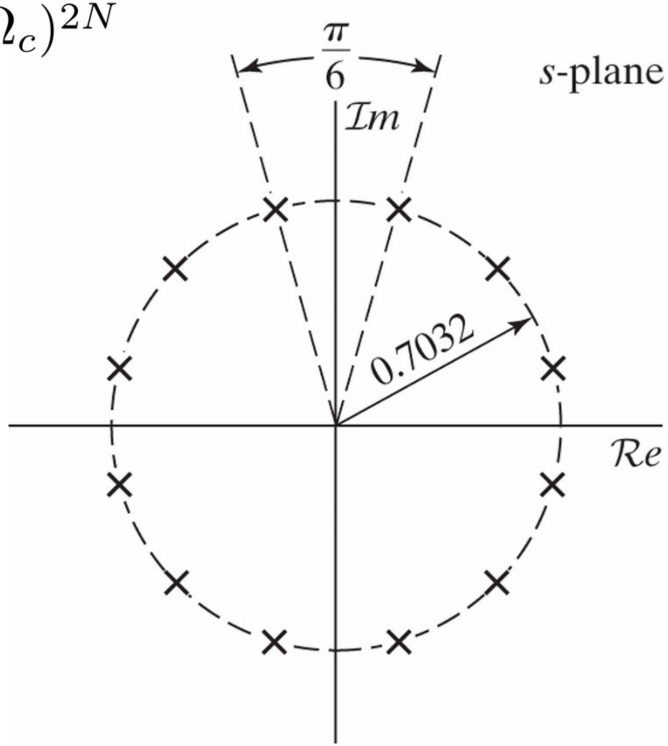
- Let $N = 6$ and $\Omega_c = 0.7032$ to exactly meet the passband specifications
  - Stopband specification exceeded ➔ margin for aliasing

Impulse invariance with Butterworth filter

- Rewrite the magnitude-squared function

$$H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}$$

  - The system function has 12 poles
- To have a stable filter, $H_c(s)$ should have three pole pairs in the left half of s-plane



Impulse invariance with Butterworth filter

- With three pole pairs

$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

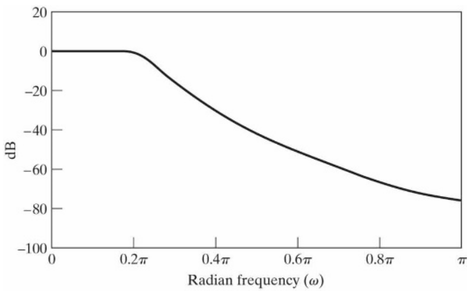
- After partial fraction, use the transformation

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

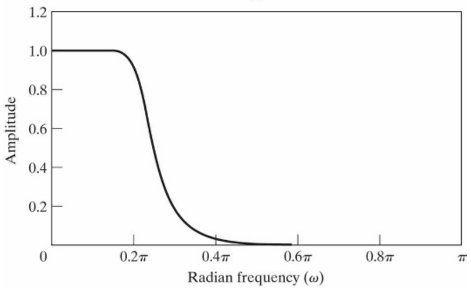
- Final discrete-time filter

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$

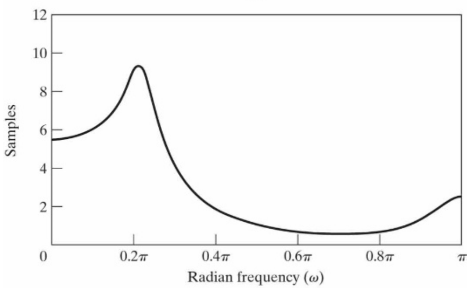
Impulse invariance with Butterworth filter



(a)



(b)



(c)

Discrete-time IIR filter design – bilinear transformation

- Continuous-time (analog) filter designed using s-plane (Laplace transform)

$$s = \sigma + j\Omega$$

$$H_c(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$H_c(j\Omega) = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt$$

$$z = re^{-j\omega}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

- Mapping between s-plane and z-plane

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

Rational behind bilinear transformation

- Recall $H_c(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

$$z = e^{sT}$$

T: numerical integration step size of the trapezoidal rule

$$s = \frac{1}{T}\ln(z)$$

Series based on area hyperbolic tangent function

$$= \frac{2}{T}\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right]$$

$$\approx \frac{2}{T}\frac{z-1}{z+1}$$

$$= \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Bilinear transformation - concept

- Given $s = \sigma + j\Omega$

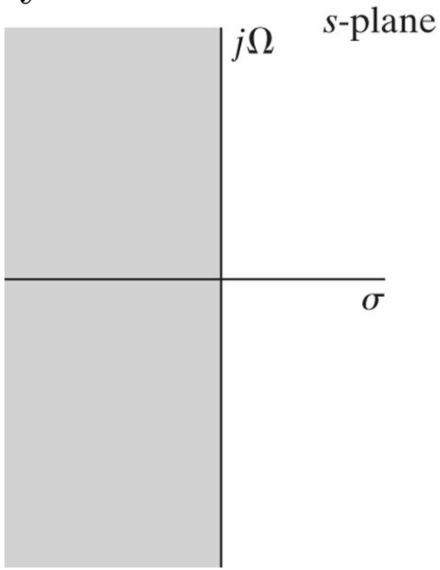
$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2}$$

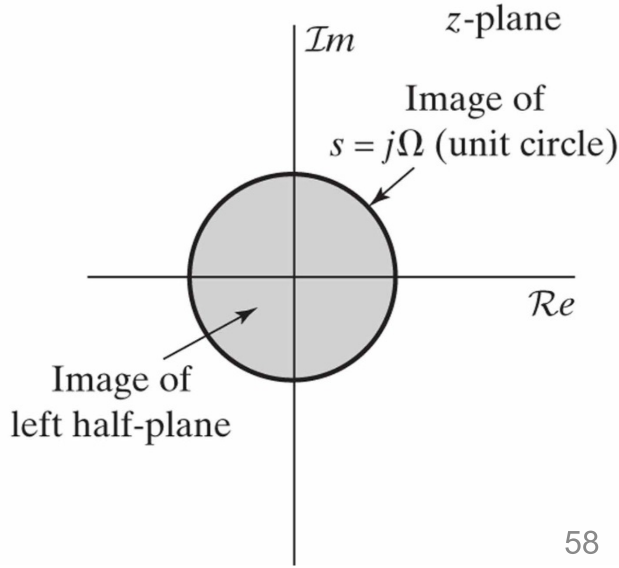
  - If $\sigma < 0, \quad |z| < 1 \text{ for any } \Omega$
  - If $\sigma > 0, \quad |z| > 1 \text{ for any } \Omega$

- Given $s = j\Omega$

$$z = \frac{1+j\Omega T_d/2}{1-j\Omega T_d/2}$$

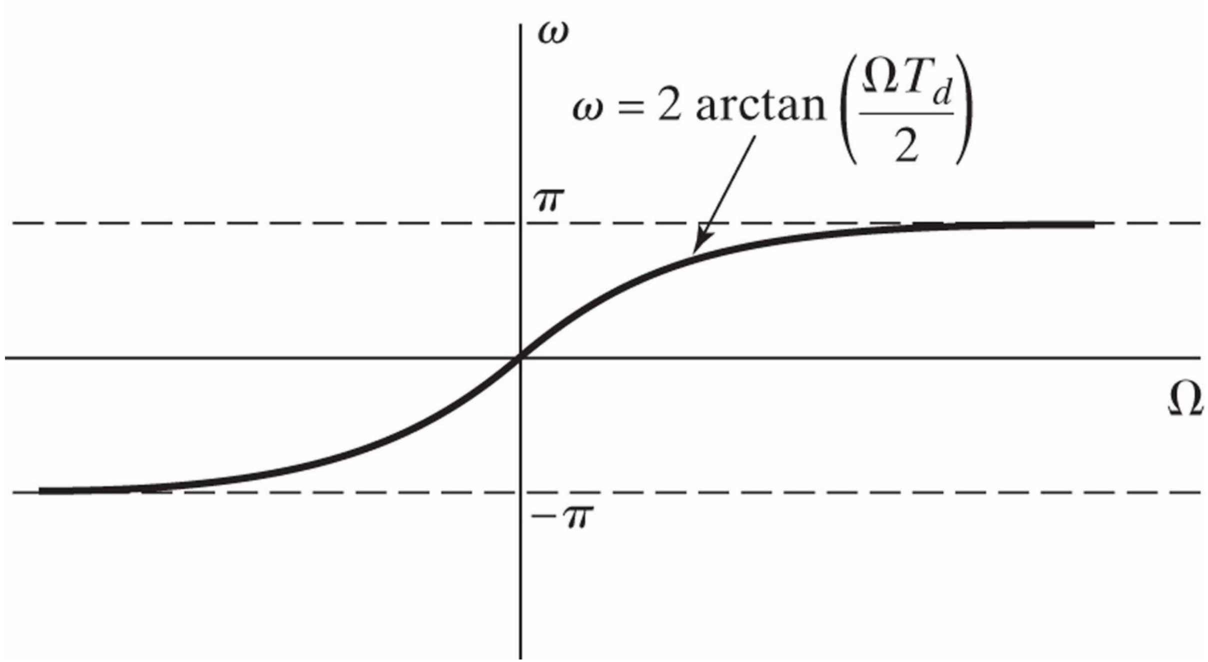
➔ |z|=1 for any s





Bilinear transformation – frequency relationship

- $\Omega = \frac{2}{T_d}\tan(\omega/2), \quad \omega = 2\arctan(\Omega T_d/2)$

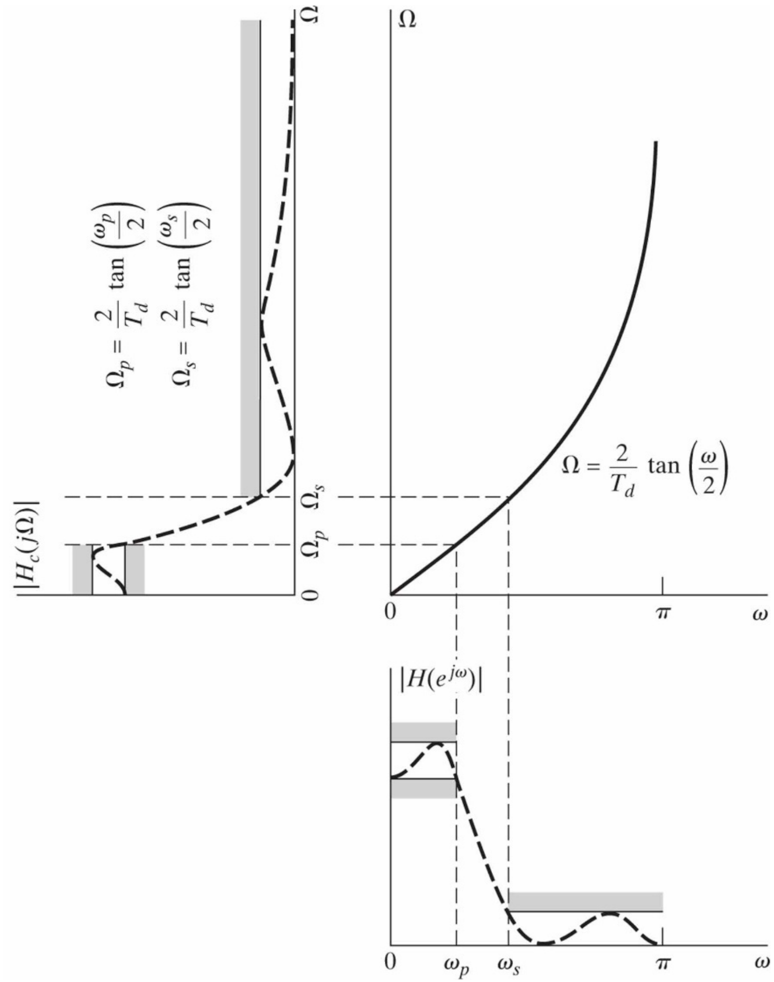


➡ Frequency warping

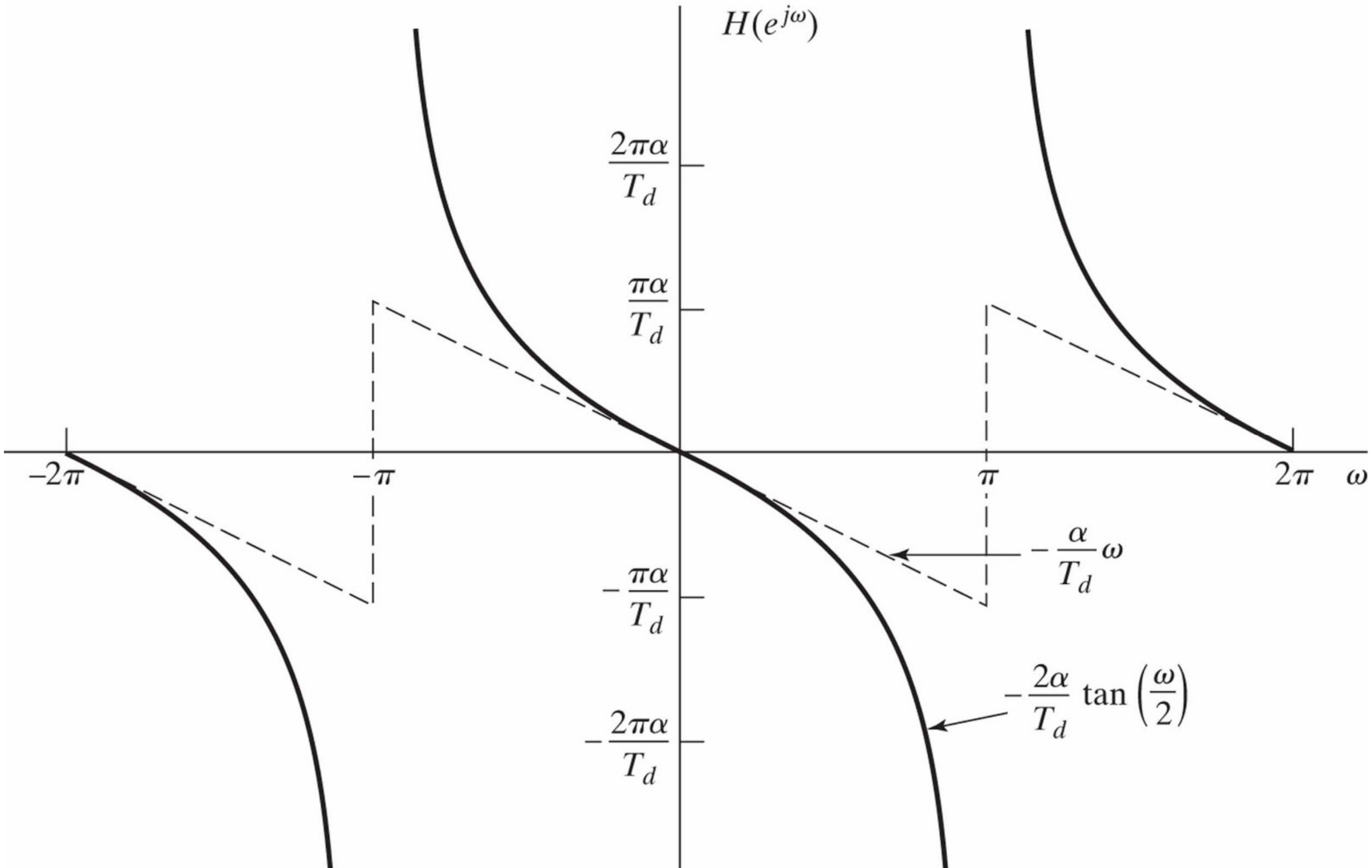
Bilinear transformation

- No problem of aliasing compared to impulse invariance method
  - Good for highpass filter design

- There exists the nonlinear compression of the frequency axis
  - Suitable for piecewise-constant magnitude response filters
  - Linear phase analog filters may lose linear phase property after transformation



Effect on phase response



Impulse invariance vs. bilinear transformation

- Bilinear transformation
  - No aliasing effect
  - Not good for preserving phase response

- Impulse invariance
  - Aliasing happens due to sampling
  - Possible to preserve linear phase of analog filter
    - Suitable to differentiator that requires linear phase

Bilinear transformation with Butterworth filter

- Specifications: $0.89125 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi$

$$|H(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi$$

- Transformed analog specifications

$$0.89125 \leq |H_c(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_c(j\Omega)| \leq 0.17783, \quad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \leq |\Omega| \leq \infty$$

- Due to monotonicity of Butterworth filter

$$|H_c(j2\tan(0.1\pi))| \geq 0.89125, \quad |H_c(j2\tan(0.15\pi))| \leq 017783$$

Bilinear transformation with Butterworth filter

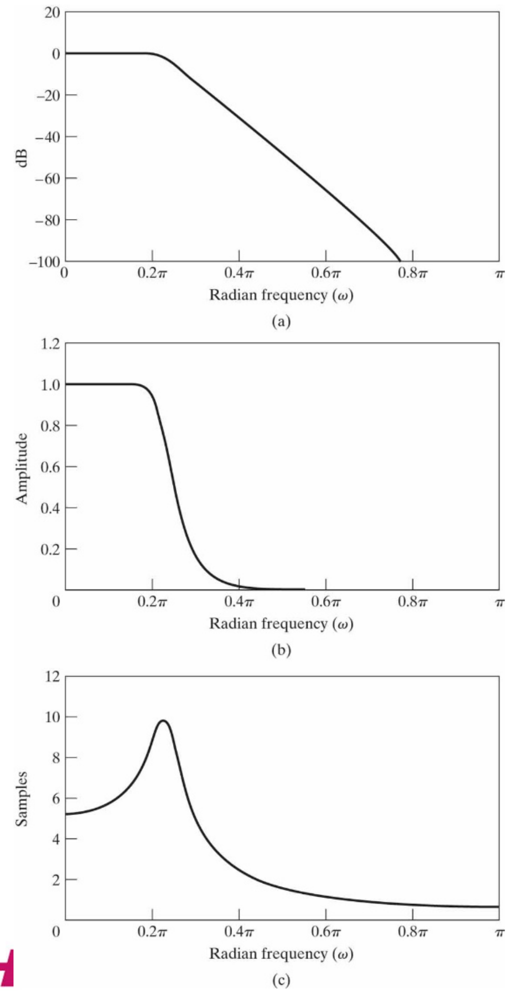
- Using similar approach as in impulse invariance method, we get *N*=5.305
- Let $N = 6,\ \Omega_c = 0.766$, which now satisfies the stopband specification

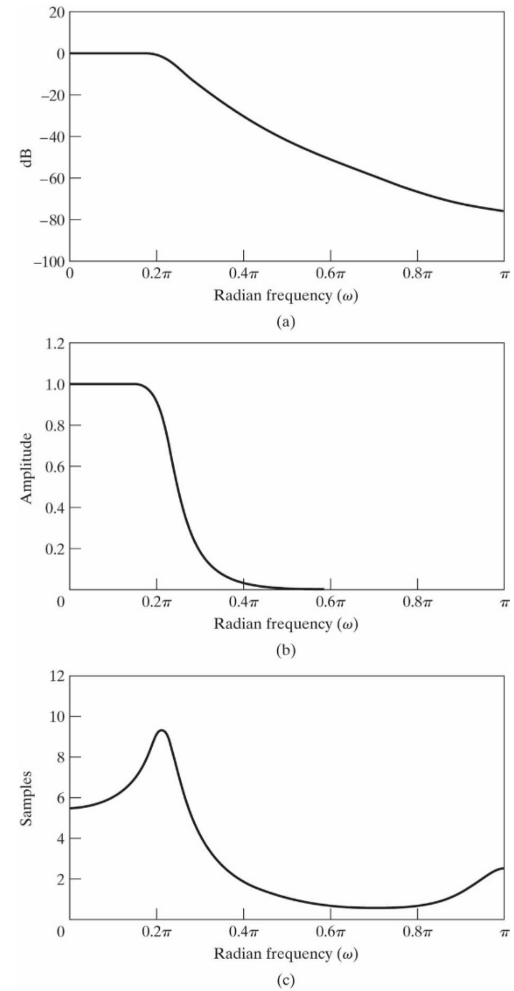
$$|H_c(j2\tan(0.15\pi))| \leq 017783$$

- This is reasonable for bilinear transformation due to lack of aliasing
  - Possible to have the desired stopband edge
- Derive stable system function $H_c(s)$ and apply bilinear transformation

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

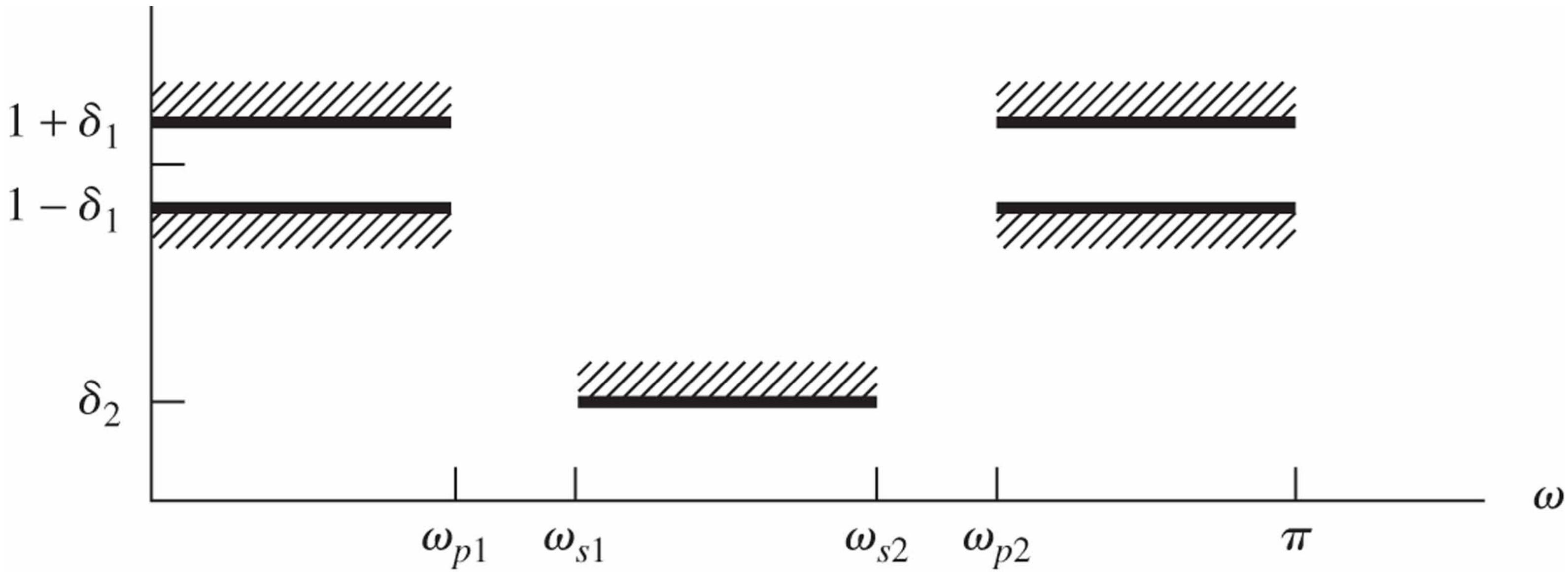
Impulse invariance vs. bilinear transformation





Frequency transformation of lowpass IIR filter

- So far, we have focused on lowpass IIR filter
- How can we implement general bandpass (multiband) filters?



Possible approaches

- Transform from analog multiband filter
  - Acceptable only with bilinear transformation
  - Impulse invariance suffers from aliasing
    ➔ Hard to implement highpass (or multiband) filters
- Transform from discrete-time lowpass filter
  - Works for both impulse invariance and bilinear transformation

Transformation table

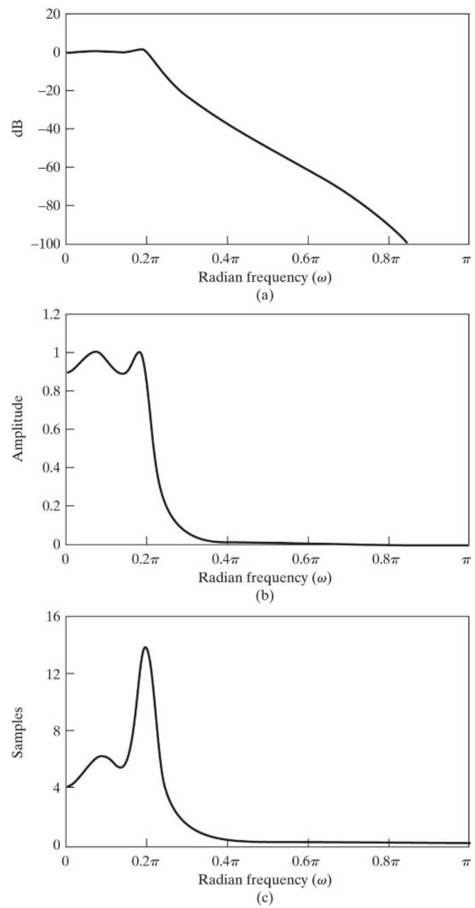
Z-plane of prototype lowpass filter

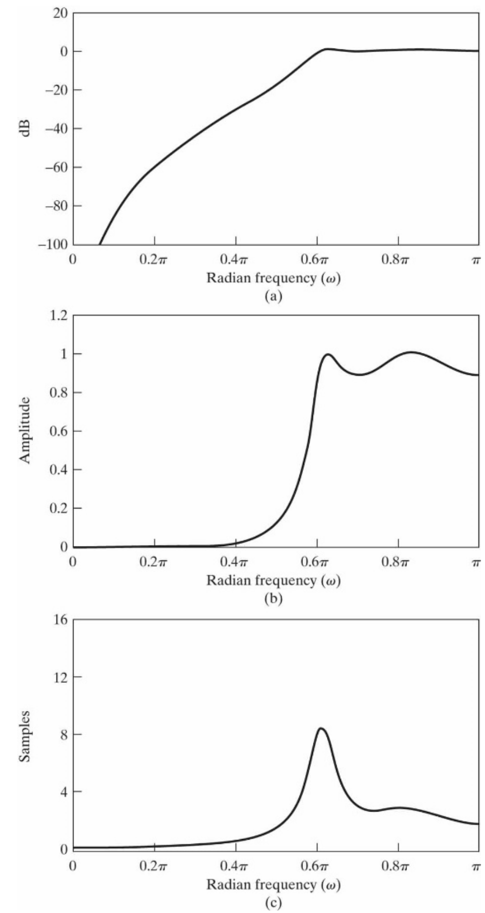
z-plane of desired filter

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency

Lowpass to highpass filter transformation





IIR filter with linear phase

- IIR filters generally have nonlinear phases
- Possible to have linear phase IIR filters for non real-time applications

$$x[n] \longrightarrow \boxed{\text{IIR } \; h[n]} \longrightarrow z[n]$$

$$z[-n] \longrightarrow \boxed{\text{IIR } \; h[n]} \longrightarrow w[n]$$

$$y[n] = w[-n]$$

Frequency-domain analysis

- $Z(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

  $$z[-n] \stackrel{\mathcal{DTFT}}{\longleftrightarrow} Z^*(e^{j\omega})$$

  $$\begin{aligned} W(e^{j\omega}) &= H(e^{j\omega})Z^*(e^{j\omega}) \\ &= H(e^{j\omega})H^*(e^{j\omega})X^*(e^{j\omega}) \\ &= |H(e^{j\omega})|^2 X^*(e^{j\omega}) \end{aligned}$$

- Since $y[n] = w[-n]$

  $$Y(e^{j\omega}) = W^*(e^{j\omega}) = |H(e^{j\omega})|^2 X(e^{j\omega})$$

  Real number → no phase distortion at all!

Matlab example

```
% Linear phase IIR filter example from Mathworks.com
fs = 100;
t = 0:1/fs:1;
x = sin(2*pi*t*3)+.25*sin(2*pi*t*40);

b = ones(1,10)/10;              % 10 point averaging filter
y = filtfilt(b,1,x);            % Noncausal filtering
yy = filter(b,1,x);             % Normal filtering
plot(t,x,t,y,'--',t,yy,':')
```