Digital Signal Processing

POSTECH
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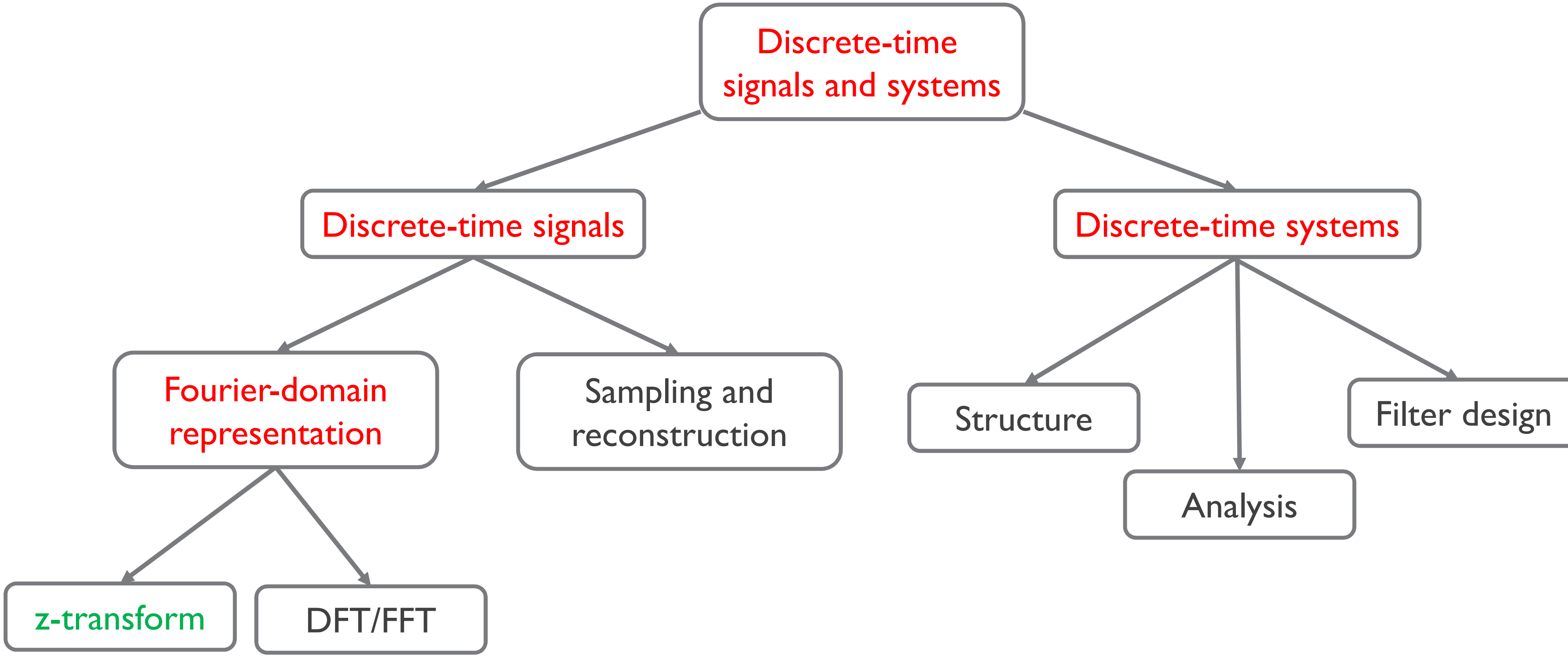
Course at glance



Inverse z-Transform

Definition of inverse z-transform

- The inverse z-transform is defined as

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C represents a closed contour within the ROC of the z-transform

➡ Hard to evaluated!

- Take different approaches in practice
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

Inspection method

- Nothing but memorizing z-transform pairs or use lookup tables (e.g., Table 3.1)

- Frequently arising pairs

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

- Memorizing these forms will significantly reduce the time to solve problems!

Partial fraction expansion

- Hard to explain in words (check section 3.3.2) but concept is simple
- Consider

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}$$

Right-sided sequence

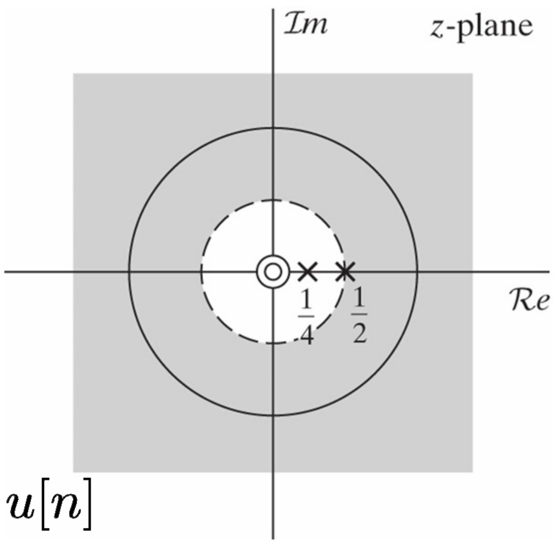
- Expand as

$$X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

where

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=1/4} = -1$$

$$A_2 = \left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=1/2} = 2$$



- By inspection method, $x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$

Another example on partial fraction

- Find the inverse of

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}, \quad |z| > 1$$

- Both numerator and denominator are the second-order
  ➔ There is a constant in X(z)
- By long division (or direct division), $X(z) = 2 + \dfrac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$
- Therefore,

$$X(z) = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

  where $A_1 = -9, \; A_2 = 8$
- When $|z| > 1$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power series expansion

- Expand $X(z)$ as a sum of polynomials z
- Example 1:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

From inspection method, $x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$

- Example 2: $X(z) = \log(1 + az^{-1}),\ \ |z| > |a|$

Use Taylor series expansion for $\log(1+x)$ with $|x| < 1$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \xleftrightarrow{z} x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

z-Transform Properties

z-transform properties - preliminaries

- Some definition

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z), \quad \text{ROC} = R_x$$

A set of values of z in ROC

- Consider two sequences

$$x_1[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z), \quad \text{ROC} = R_{x_1}$$

$$x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_2(z), \quad \text{ROC} = R_{x_2}$$

Useful z-transform properties

- Linearity
- Time shifting
- Multiplication by an exponential sequence
- Differentiation of X(z)
- Conjugation of a complex sequence
- Time reversal
- Convolution of sequences

- Major difference from DTFT

  ➡ Need to carefully consider ROC

Table of z-transform properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

How to use z-transform properties?

- Show

$$X(z) = \log(1 + az^{-1}),\ \ |z| > |a| \overset{\mathcal{Z}}{\longleftrightarrow} x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

z-Transform and LTI Systems

z-Transform and LTI systems - Preview

- z-transform and its properties are very useful tools for discrete-time system analysis

- Brief introduction of LTI system analysis using z-transform
  - Will be discussed in detail later

LTI system analysis using z-transform

- Using the convolution property

$$y[n] = h[n] * x[n] \overset{z}{\longleftrightarrow} Y(z) = H(z)X(z)$$

System function of LTI system

Assume $|a| < 1$

- Easy to compute output of LTI system

$$h[n] = a^n u[n],\ x[n] = Au[n],\ y[n] = h[n] * x[n] = ?$$

$$H(z) = \frac{1}{1 - az^{-1}},\ |z| > |a| \qquad X(z) = \frac{A}{1 - z^{-1}},\ |z| > 1$$

$$Y(z) = \frac{Az^2}{(z-a)(z-1)} = \frac{A}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right),\ |z| > 1$$

$$y[n] = \frac{A}{1-a}(1 - a^{n+1})u[n]$$

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z-transform for difference equations

- z-transform is particularly useful for LTE systems with difference equations

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

- Due to linearity and time-shift properties

$$Y(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z) + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z)$$

$$\Rightarrow Y(z) = \left(\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}\right)X(z)$$

→ $H(z)$!

Example

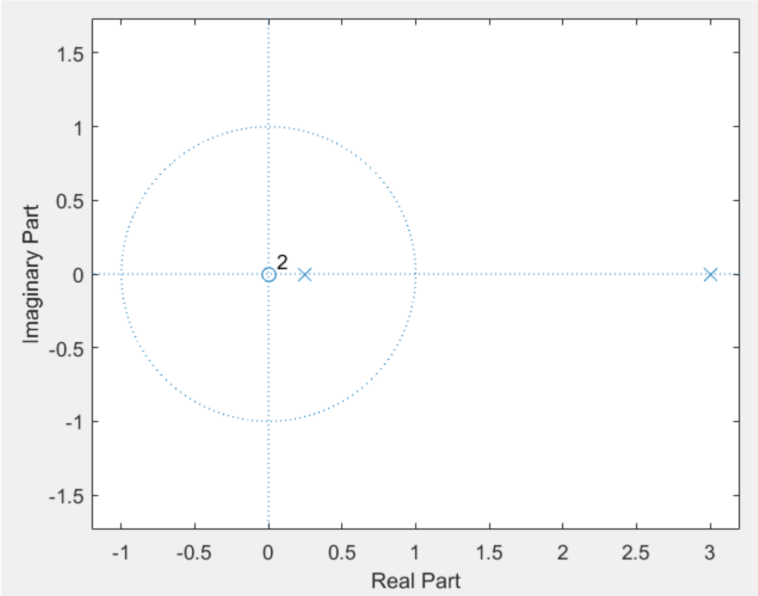
- Let $y[n] = \frac{13}{4}y[n-1] - \frac{3}{4}y[n-2] + x[n]$

- z-transform gives

$$Y(z) = \frac{13}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{13}{4}z^{-1} + \frac{3}{4}z^{-2}}$$

$$= \frac{z^2}{z^2 - \frac{13}{4}z + \frac{3}{4}} = \frac{z^2}{\left(z - \frac{1}{4}\right)(z-3)}$$



Example

- Using partial fraction expansion

$$H(z) = -\frac{1}{11}\frac{z}{z-\frac{1}{4}} + \frac{12}{11}\frac{z}{z-3}$$

- Three possibilities for ROC
  - $|z| < \frac{1}{4}$
  - $\frac{1}{4} < |z| < 3$
  - $|z| > 3$

Example

- If $|z| < \frac{1}{4}$

- Impulse response becomes

$$h[n] = \frac{1}{11}\left(\frac{1}{4}\right)^n u[-n-1] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Left-sided sequence

- BIBO stable? No! $\lim_{n \to -\infty} |h[n]| = \infty$

Example

- If $\frac{1}{4} < |z| < 3$
- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Two-sided sequence
- BIBO stable? Yes! $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Example

- If $|z| > 3$

- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{11}(3)^n u[n]$$

- Causal? Yes!

- BIBO stable? No!

Stability and causality

- Stability requires ROC to include unit circle $|z| = 1$
  - Proof using triangle inequality $|a+b| \leq |a| + |b|$

$|z| = 1$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

For BIBO stability

- Causality requires ROC to satisfy $|z| > |p_N|$

Largest pole

- If the system to be stable AND causal
  ➔ $|p_N| < 1$
  ➔ All poles must be located within unit circle

Basic Filter Analysis Using z-Transform

Notch filters (bandstop filter with narrow stopband)

- Want to get rid of frequency component at ω_0

- z-transform representation of general notch filters

$$H_{\text{notch}}(z) = \frac{G(z - e^{j\omega_0})(z - e^{-j\omega_0})}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}$$

- Clearly, $H_{\text{notch}}(e^{j\omega_0}) = H_{\text{notch}}(e^{-j\omega_0}) = 0$

- The difference equation for notch filter

$$y[n] = 2r\cos(\omega_0)y[n-1] - r^2 y[n-2] + Gx[n] - G2\cos(\omega_0)x[n-1] + Gx[n-2]$$

Matlab example of notch filter

```
%Shows how simple difference equation can remove a tone
%that's corrupting a speech utterance.
clf
clear all
%these commands read in the speech file: need getspeech.m
datar=getspeech('woman_voice.wav');
Fs=12500;
%plot data to cut off silence
plot(datar)
[d,dsize]=size(datar);
input('play back utterance at 12.5 KHz sampling rate');
soundsc(datar,Fs)
input('add tone at 3.125 KHz to utterance and play back');
omega_noise=pi/2;
nc=1:dsize;
x=datar+500*cos(omega_noise*nc);
Figure(2)
plot(x)
soundsc(x,Fs)
%define coefficients for second-order notch filter
r=0.95;
omega0=pi/2;
input('run tone corrupted speech through simple second order difference equation');
y(1)=0; y(2)=0;
for n=3:dsize
y(n)=2*r*cos(omega0)*y(n-1)-r^2*y(n-2)+x(n)-2*cos(omega0)*x(n-1)+x(n-2);
end
input('play back output of difference eqn.');
soundsc(y,Fs)
```

All-pass filters

- Mathematical preliminary
  - Let $c = a + jb = |c|e^{j\angle c}$
  - Note $\dfrac{c}{c^*} = \dfrac{|c|e^{j\angle c}}{|c|e^{-j\angle c}} = 1e^{j2\angle c}$ ➔ $\left|\dfrac{c}{c^*}\right| = 1$
- Consider the system with single pole at $z = p$ and a single zero at $z = \dfrac{1}{p^*}$

$$H(z) = G\frac{z - \frac{1}{p^*}}{z - p}$$

- Frequency response becomes

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = G\frac{e^{j\omega} - \frac{1}{p^*}}{e^{j\omega} - p} = -\frac{G}{e^{j\omega}p}\frac{c}{c^*}, \text{ where } c = e^{j\omega} - \frac{1}{p^*}$$

All-pass filters

- The amplitude of frequency response

$$|H(e^{j\omega})| = \left|\frac{G}{p}\right| = \frac{|G|}{|p|}$$

does not depend on ω ➔ all-pass filter!

- An all-pass filter can be used to stabilize an unstable system without affecting the magnitude of the frequency response
  - Example: assume $|p| < 1$

$$\boxed{\frac{H'(z)}{z - \frac{1}{p^*}}} \times \frac{z - \frac{1}{p^*}}{z - p}$$

Everything except $z - 1/p^*$

Original system

The original pole $z = 1/p^*$ outside unit circle

The new pole z=p inside unit circle

➔ The overall system becomes stable!

➔ Magnitude is unaffected

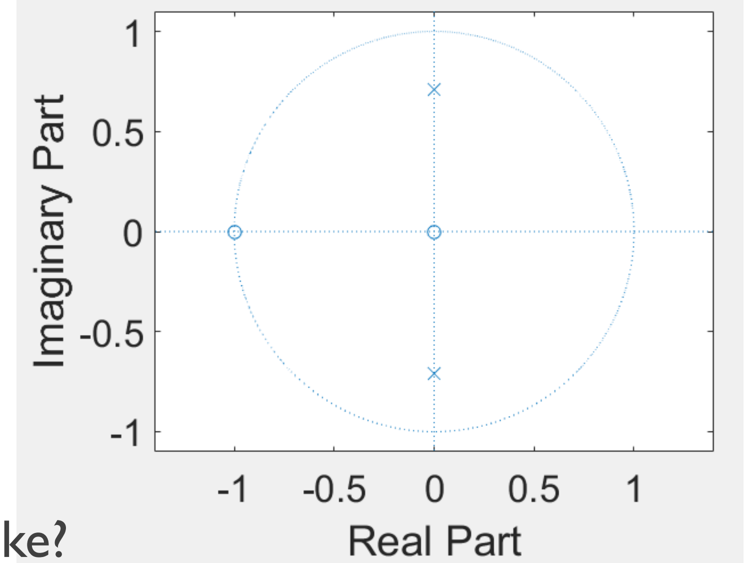
Using poles and zeros to design filters

- Through judicious positioning of zeros and poles
  - Emphasize "desired" frequency bands
  - De-emphasize other frequency bands

- Example:

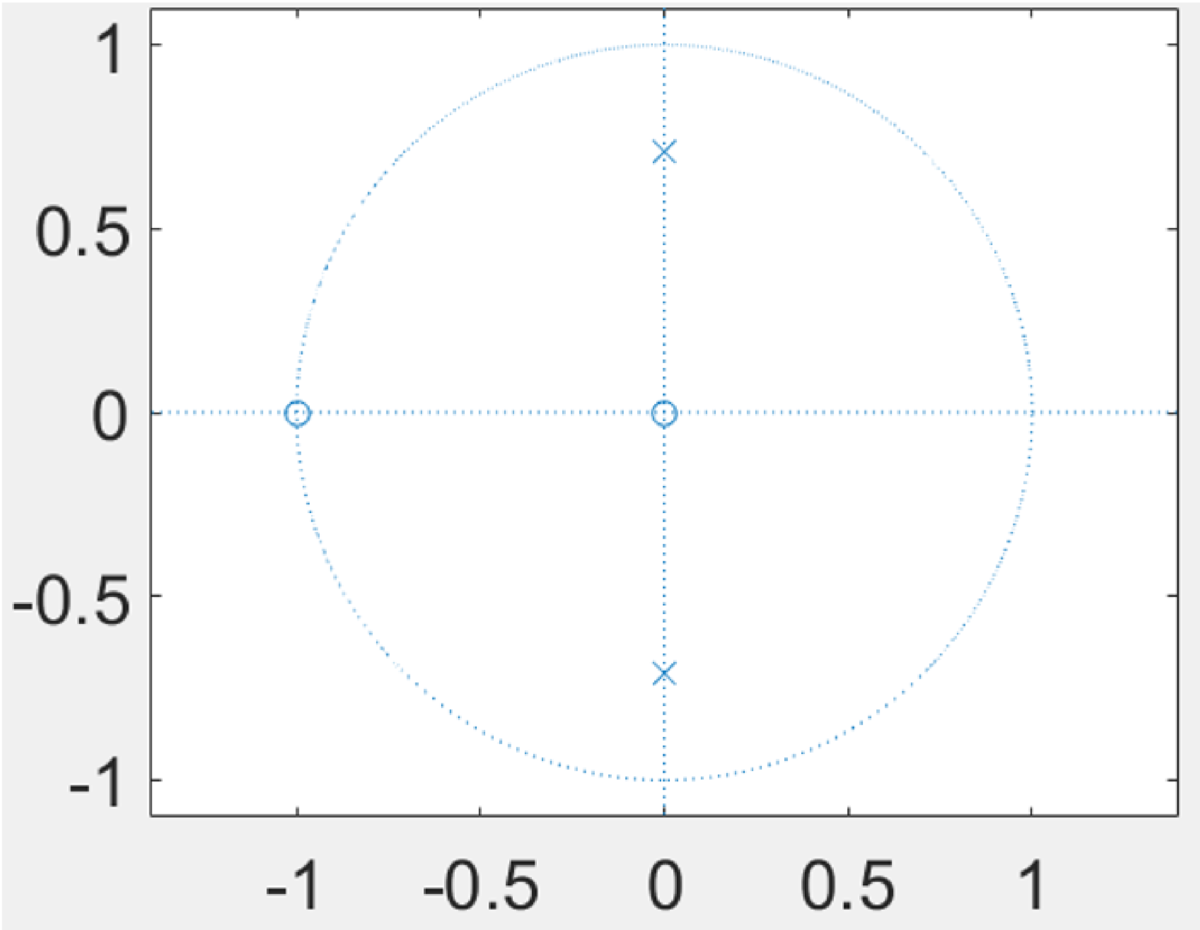
$$y[n] = -\frac{1}{2}y[n-2] + x[n] + x[n-1]$$

$$H(z) = \frac{1+z^{-1}}{1+\frac{1}{2}z^{-2}} = \frac{z(z+1)}{\left(z - j\frac{1}{\sqrt{2}}\right)\left(z + j\frac{1}{\sqrt{2}}\right)}$$



How the frequency response $H(e^{j\omega})$ does look like?

Graphical evaluation of magnitude



$$|H(z)| = \frac{|z||(z+1)|}{\left|z - j\frac{1}{\sqrt{2}}\right| \left|z + j\frac{1}{\sqrt{2}}\right|}$$

Homework

- Problems in textbook: 3.23, 3.29, 3.30, 3.36, 3.45
  - Solution uploaded on the webpage

MATLAB Programming

Pole-zero plot for z-transform

- Consider

$$X(z) = \frac{p_0 + p_1 z^{-1} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_N z^{-N}}$$

- 'zplane(num,den)' gives pole-zero plot
  - 'num' and 'den' are row vectors
- 'zplane(zeros,poles)' also gives pole-zero plot
  - 'zeros' and 'poles' are column vectors
- '[z,p,k]=tf2zp(num,den)' gives poles, zeros, and gain constant

- Also, check out 'zp2tf' function