Digital Signal Processing

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Fourier Transform Representations

Why frequency response important?

- A broad class of signals can be represented as $x[n] = \sum_k \alpha_k e^{j\omega_k n}$
  - Example: $A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$
- The output of LTI system is

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

- If we know $H(e^{j\omega_k})$ for all k, the output of the LTI system can be easily computed

Frequency-domain representation

- Fourier transform pair
  - Many sequences can be represented as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ → Inverse Fourier transform

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ → (Discrete-time) Fourier transform

  - Note that x[n] is represented as a superposition of infinitesimally small complex sinusoids

$$\frac{1}{2\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

with ω ranging over an interval of length 2π

Sufficient condition for Fourier transform

- Fourier transform should be finite to exist

$$|X(e^{j\omega})| < \infty \quad \text{for all } \omega$$

- **Sufficient condition** for the convergence of $X(e^{j\omega})$

Not necessary

$$\begin{aligned} |X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \\ &\le \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \\ &\le \sum_{n=-\infty}^{\infty} |x[n]| < \infty \end{aligned}$$

➡ All absolutely summable sequences have Fourier transform

Not absolutely summable sequences

- Consider square summable $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

- For such sequences, mean-square convergence exists

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0$$

where

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

Sequences neither absolutely nor square summable

- Constant sequence: x[n]=1 for all n

Impulse train with period 2π

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

- Complex exponential sequence: $x[n] = e^{j\omega_0 n}$

Shifted impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r)$$

- Generalization of exponential sequence $x[n] = \sum_k a_k e^{j\omega_k n}$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

Symmetry properties of Fourier transform

Table 2.1 in book

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Fourier transform theorems

TABLE 2.2 FOURIER TRANSFORM THEOREMS

	Sequence	Fourier Transform
	$x[n]$	$X(e^{j\omega})$
	$y[n]$	$Y(e^{j\omega})$
1.	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2.	$x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3.	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4.	$x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5.	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6.	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7.	$x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Periodic convolution

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$

Useful Fourier transform pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

Using Fourier transform theorems and pairs

- Using the tables, we can
  - Compute Fourier transform of sequences without complex computations
  - Compute inverse Fourier transform
  - Example: $\mathcal{F}\{a^n u[n-5]\} = ?$

$$x_1[n] = a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1 - ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}$$

Another example

- Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$

- To get the impulse response, set $x[n] = \delta[n]$, which gives $y[n] = h[n]$

$$h[n] - \frac{1}{2}h[n-1] = \delta[n] - \frac{1}{4}\delta[n-1]$$

- Fourier transform gives

$$H(e^{j\omega}) - \frac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

Another example

- Decompose $H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}} - \dfrac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

- By using the table

$$\left(\frac{1}{2}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}e^{-j\omega}}, \quad -\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

- h[n] becomes

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- Read Examples 2.23-2.24 in the textbook

Auto/cross-correlations of Deterministic Signals

Definitions

- Autocorrelation $c_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x^*[n-\ell]$

- Cross-correlation $c_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y^*[n-\ell]$

Conjugate for complex sequences

- Relation to convolution: missing initial fold step

$$c_{xy}[\ell] = x[\ell] * y^*[-\ell]$$
$$c_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

Pseudo-noise (PN) sequence

- Let $x[n] = \{1, 1, 1, -1, -1, 1, -1\}$

  $n = 0$

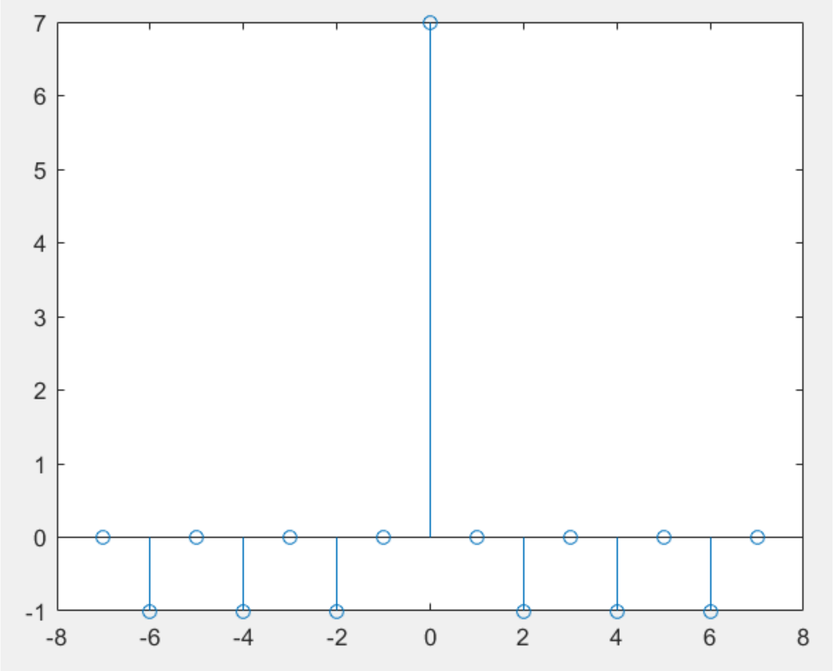
- Autocorrelation

$$c_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n] x^*[n-\ell]$$

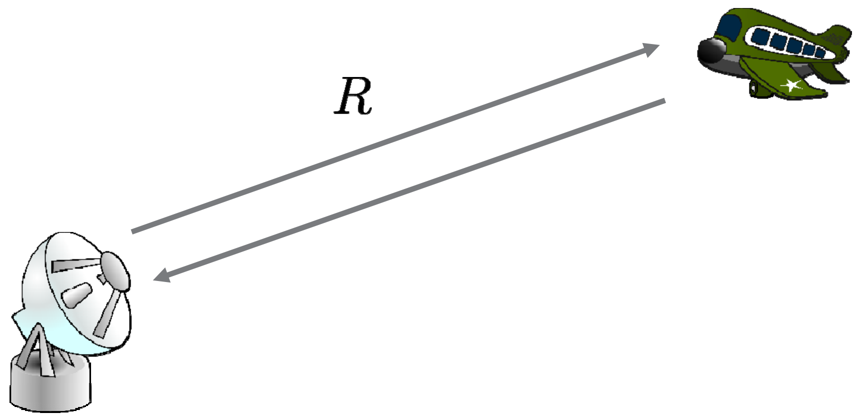
- Example of Barker code
- Sharp peak at $\ell = 0$
- Time-delay estimation in radar

$\{1, 1, 1, -1, -1, 1, -1\}$

$\{1, 1, 1, -1, -1, 1, -1\}$ →



Radar example



- Transmit pulse $S_a(t)$
- Received "echo" after reflection off object

$$y_a(t) = \Gamma S_a(t - \tau_d) + w_a(t)$$

- Γ: Unknown amplitude
- τ_d: Round-trip time-delay, $\tau_d = \frac{2R}{c}$
- $w_a(t)$: Noise

Radar example

- Sampled version $y[n] = \Gamma S_a(nT_s - \tau_d) + w[n]$
- Assume sampling high enough: $\tau_d = DT_s, \text{ where } D \text{ is integer}$

$$S_a(nT_s - DT_s) = S_a((n-D)T_s) = s[n-D], \text{ where } s[n] = S_a(nT_s)$$

- Discrete-time (DT) model

$$y[n] = \Gamma s[n-D] + w[n] = s[n] * \Gamma\delta[n-D] + w[n]$$

  - Without noise, it can be modeled as an LTI system

$$x[n] \longrightarrow \boxed{h[n] = \Gamma\delta[n-D]} \longrightarrow y[n]$$

- Use cross-correlation to estimate D ➔ $R = \dfrac{cDT_s}{2}$

Radar example

- $$c_{ys}[\ell] = \sum_{n=-\infty}^{\infty} y[n]s[n-\ell]$$

Assume s[n] is a real signal, e.g., Barker code

$$= \sum_{n=-\infty}^{\infty} \left(\Gamma s[n-D] + w[n]\right) s[n-\ell]$$

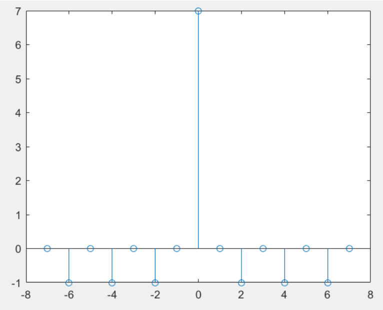
$$= \sum_{n=-\infty}^{\infty} \Gamma s[n-D]s[n-\ell] + \sum_{n=-\infty}^{\infty} w[n]s[n-\ell]$$

$$n' = n - D$$

$$= \Gamma \sum_{n'=-\infty}^{\infty} s[n']s[n'-(\ell - D)] + c_{ws}[\ell]$$

$$= c_{ss}[\ell - D] + c_{ws}[\ell]$$

Peak at $\ell = D$



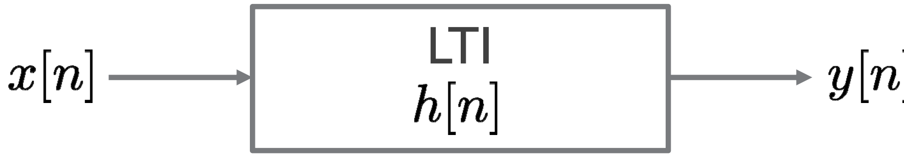
Properties of autocorrelation sequences

- Three main properties of autocorrelation sequence

$$c_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

- $c_{xx}[-\ell] = c_{xx}^*[\ell]$
- $|c_{xx}[\ell]| \leq c_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \text{energy}$
- $\sum_{\ell=-\infty}^{\infty} c_{xx}[\ell] e^{-j\omega\ell} \geq 0, \text{ for all } \omega$

Input/output relationships for LTI system



- $c_{yx}[\ell] = c_{xx}[\ell] * h[\ell]$
- $c_{yy}[\ell] = c_{xx}[\ell] * c_{hh}[\ell]$

Try to prove these

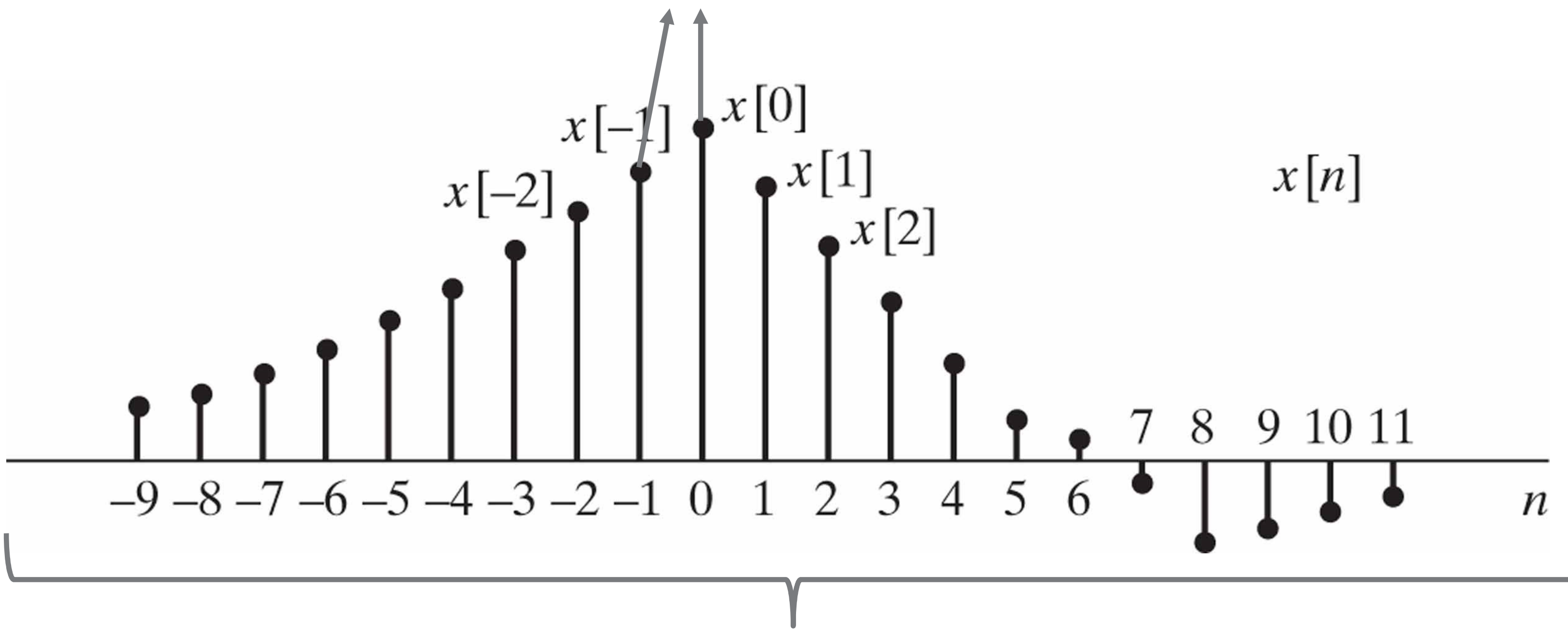
Discrete-Time Random Signals

Why random?

- So far, all signals are assumed to be deterministic
  - Sequence is uniquely determined by a mathematical expression
- In many situations, the processes that generate signals are so complex
  - Modeling water flow, wind turbulence, etc
- Modeling a signal as a random process analytically useful

Discrete-time random signals

Each sample is a random variable
following a certain probability density function



Collection of random variables is referred to as a random process

Random process

- Hard to completely describe a random process
  - Need to know individual and joint probability distributions of all random variables

- Many cases, descriptions in terms of *averages* are useful
  - Mean / Autocorrelation / Autocovariance

- Let x_n represent the random variable of the sequence x[n] at time n
  - Precisely, x_n is a random variable and x[n] is a specific realization of x_n at time n
  - It is not necessary to distinguish them in this course.
  - We will use x_n and $x[n]$ interchangeably

- **Focus on real signals**

Wide-sense stationary (WSS) random signals

- The mean of random process x[n] at time n is

$$m_x[n] = \mathcal{E}\{x[n]\}$$

- The autocorrelation of x[n] is

$$\phi_{xx}[n, n+m] = \mathcal{E}\left\{x[n]x[n+m]\right\}$$

- If x[n] is wide-sense stationary random process, then

$$m_x[n] = m_x, \quad \text{for all } n$$

$$\phi_{xx}[n, n+m] = \phi_{xx}[m], \quad \text{for all } n \text{ and } m$$

Input-output relation of LTI systems

- Recall a LTI system $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
- With WSS input, the mean of output becomes

Constant!

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\} = \boxed{m_x \sum_{k=-\infty}^{\infty} h[k]}$$

  - Alternative expression

$$m_y = H(e^{j0})m_x$$

Input-output relation of LTI systems

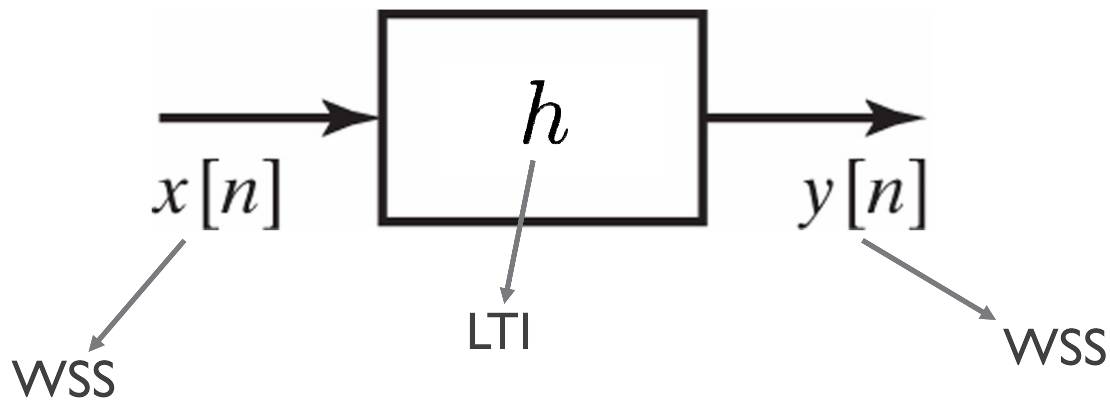
- Autocorrelation of output is

$$
\begin{aligned}
\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\} \\
&= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\left\{x[n-k]x[n+m-r]\right\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] \\
&= \phi_{yy}[m]
\end{aligned}
$$

WSS input

Not a function of *n*!

Input-output relation of LTI systems



Further look on output autocorrelation

- Recall $\phi_{yy}[m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r]$
- By taking the substitution $\ell = r - k$

$$\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \sum_{k=-\infty}^{\infty} h[k]h[\ell+k] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell]$$

where

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[\ell+k] = h[\ell] * h[-\ell]$$

Convolution

Deterministic autocorrelation sequence of $h[\ell]$

Since $h[\ell]$ is assumed to be real, $c_{hh}[-\ell] = c_{hh}[\ell]$

Fourier transform of autocorrelations

- Let $\phi_{xx}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Phi_{xx}(e^{j\omega}),\ \phi_{yy}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Phi_{yy}(e^{j\omega}),\ c_{hh}[n] \overset{\mathcal{F}}{\longleftrightarrow} C_{hh}(e^{j\omega})$
- Because $\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] c_{hh}[\ell]$

$$c_{hh}[\ell] = h[n] * h[-n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2$$

we have
$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega})$$
$$= |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})$$

Power density spectrum or power spectrum density

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})d\omega$$

Note on energy/power signals

- Energy signal
  - If a sequence has a finite energy $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
  - Fourier transform of autocorrelation of energy signal
    ➔ Energy spectrum density

- Power signal
  - If a sequence has a finite average power $P = \lim_{M \to \infty} \frac{1}{2M} \sum_{n=-M}^{M} |x[n]|^2$
  - Fourier transform of autocorrelation of power signal
    ➔ Power spectrum density

Course at glance