Digital Signal Processing

POSTECH

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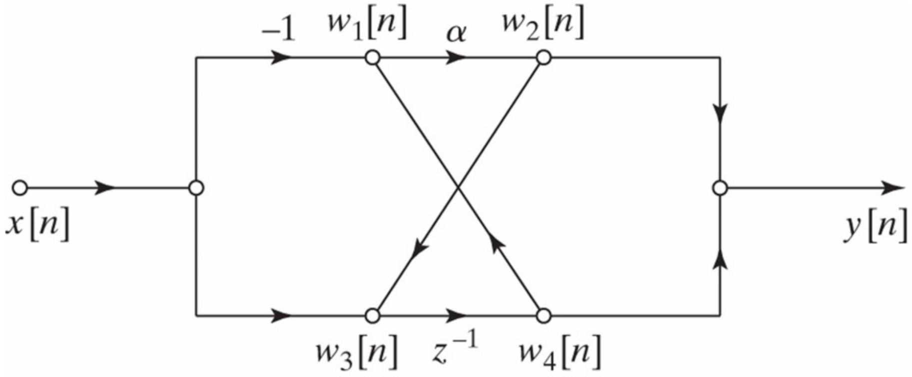
Signal Flow Graph Representation

Signal flow graph with z-transformation

- Consider the graph
  - Not a direct form
  - Cannot obtain H(z) by inspection
  - How to obtain H(z)?



- Each node is $w_1[n] = w_4[n] - x[n]$

$$w_2[n] = \alpha w_1[n]$$

$$w_3[n] = w_2[n] + x[n]$$

$$w_4[n] = w_3[n-1]$$

$$y[n] = w_2[n] + w_4[n]$$

Difficult to solve due to delay
Use z-transform!

Signal flow graph with z-transformation

- z-transform equations

$$\begin{aligned} W_1(z) &= W_4(z) - X(z) \\ W_2(z) &= \alpha W_1(z) \\ W_3(z) &= W_2(z) + X(z) \\ W_4(z) &= z^{-1} W_3(z) \\ Y(z) &= W_2(z) + W_4(z) \end{aligned}$$

- After removing some variables

$$\begin{aligned} W_2(z) &= \alpha(W_4(z) - X(z)) \\ W_4(z) &= z^{-1}(W_2(z) + X(z)) \\ Y(z) &= W_2(z) + W_4(z) \end{aligned}$$

➡

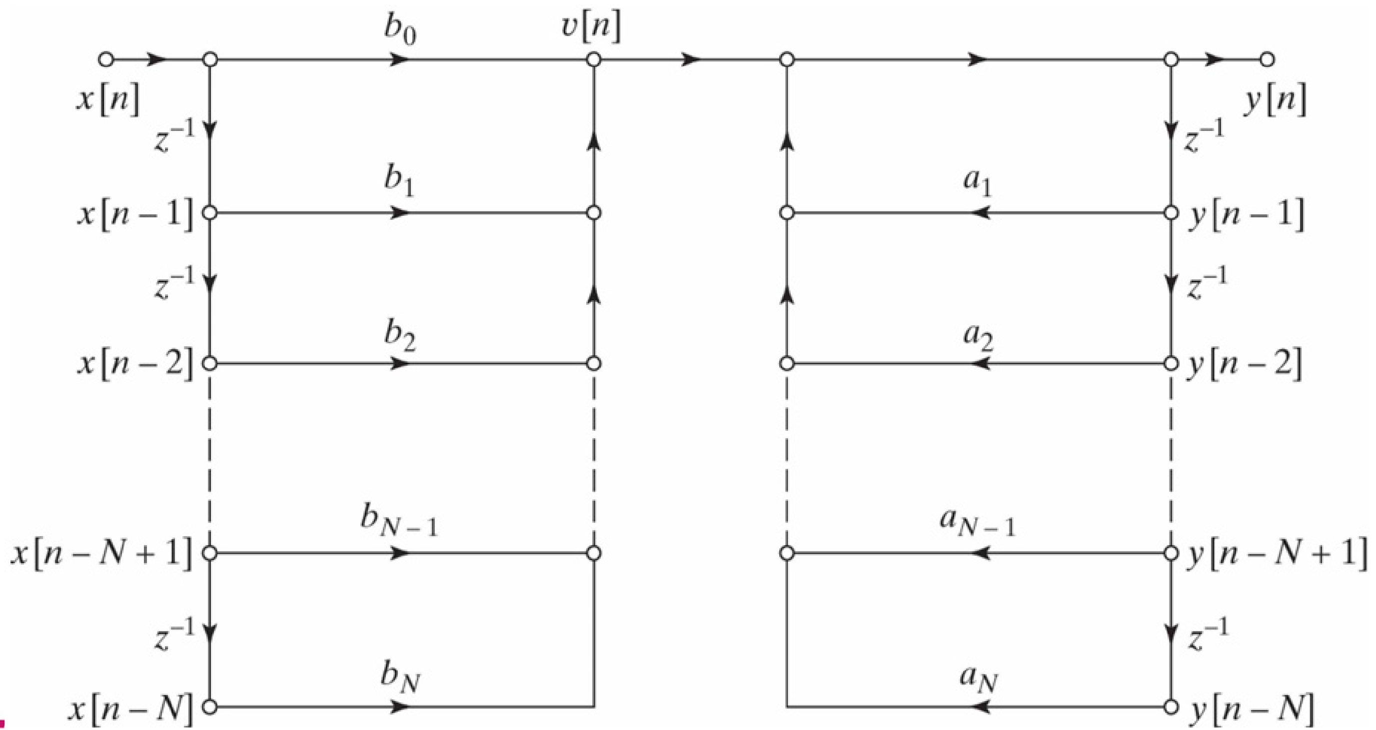
$$\begin{aligned} W_2(z) &= \frac{\alpha(z^{-1} - 1)}{1 - \alpha z^{-1}} X(z) \\ W_4(z) &= \frac{z^{-1}(1 - \alpha)}{1 - \alpha z^{-1}} X(z) \end{aligned}$$

Basic structure for IIR systems

- Similar to block-diagram representation, there can be various ways to represent a system using signal flow graph
    - Direct form I
    - Direct form II (canonic direct form)
    - Cascade form
    - Parallel form

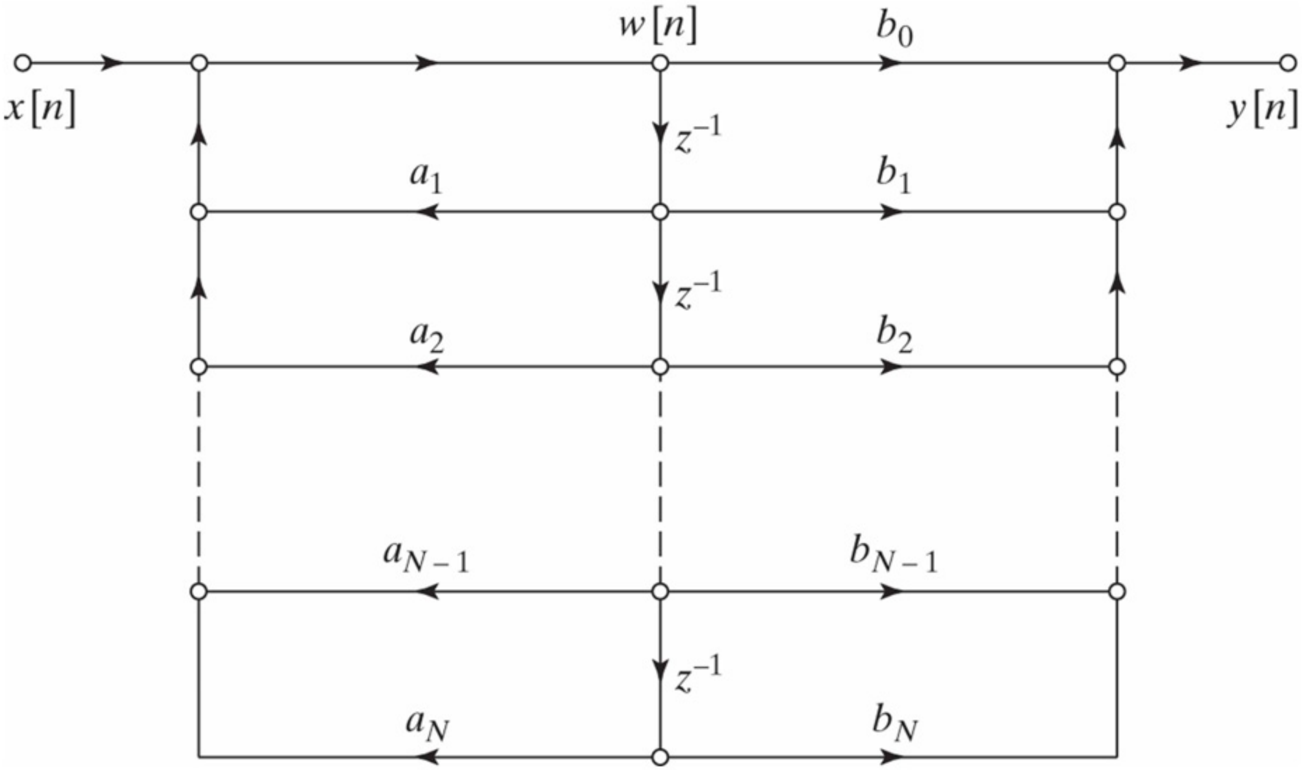
Direct form I

- Consider $y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$



Direct form II

- Consider $y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$



Cascade form

- Note $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$
- Consider the most general factorization when all coefficients are real

$$H(z) = A \underbrace{\frac{\prod_{k=1}^{M_1}(1 - f_k z^{-1})}{\prod_{k=1}^{N_1}(1 - c_k z^{-1})}}_{\text{Real poles and zeros}} \underbrace{\frac{\prod_{k=1}^{M_2}(1 - g_k z^{-1})(1 - g_k^* z^{-1})}{\prod_{k=1}^{N_2}(1 - d_k z^{-1})(1 - d_k^* z^{-1})}}_{\text{Conjugate pairs of poles and zeros}}$$

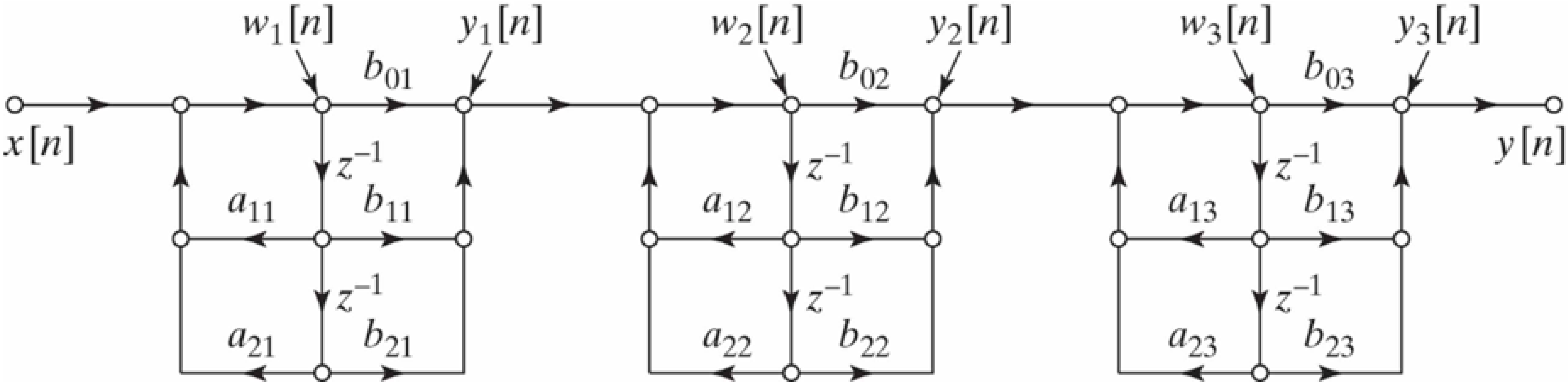
- Combine pairs of real factors and complex conjugate pairs into 2nd-order factors

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

➡ Can efficiently implement 2nd-order subsystems

Cascade form

- Example of 6-th order system



- Many ways to combine pairs of poles and zeros with the same overall system function **with infinite precision**
  - With finite precision, the results can be quite different

Parallel form

- Using partial fraction expansion

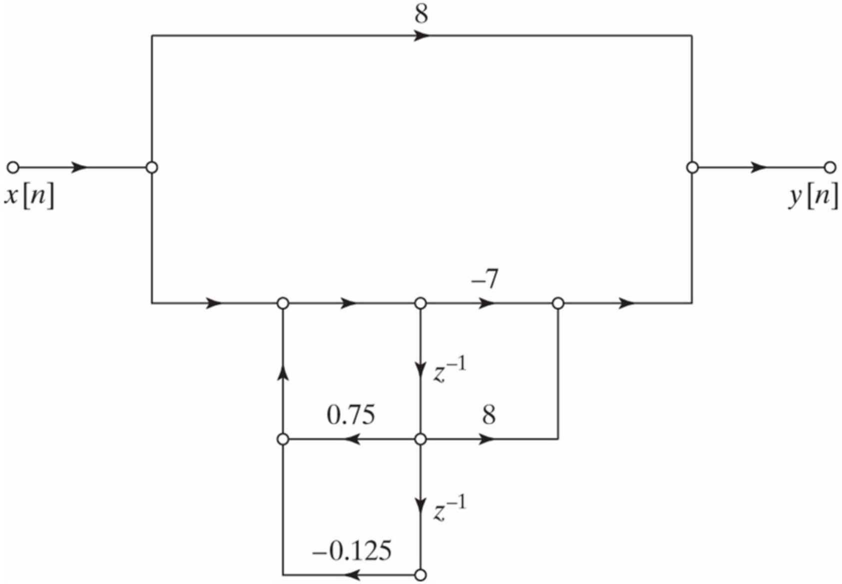
$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_1} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k(1 - e_k z^{-1})}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})}$$

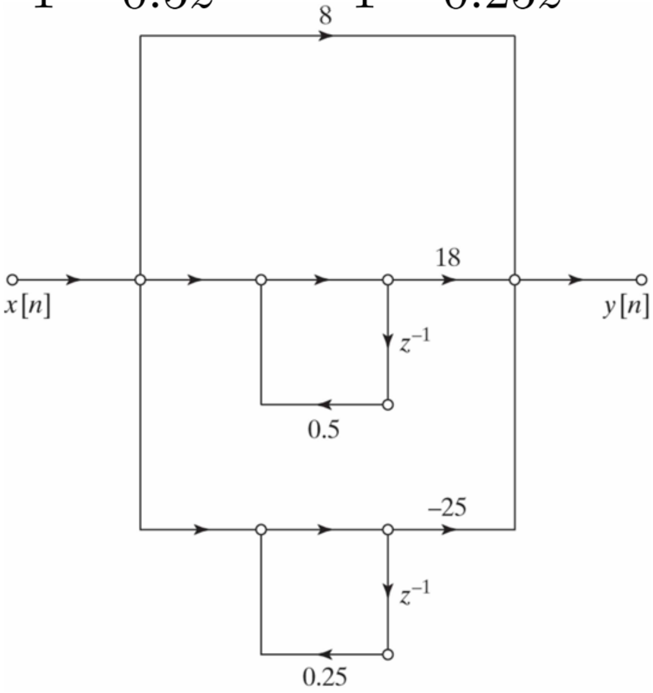
$$= \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

Parallel form example

- $H(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} = 8 + \dfrac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$

$$= 8 + \frac{18}{1-0.5z^{-1}} - \frac{25}{1-0.25z^{-1}}$$

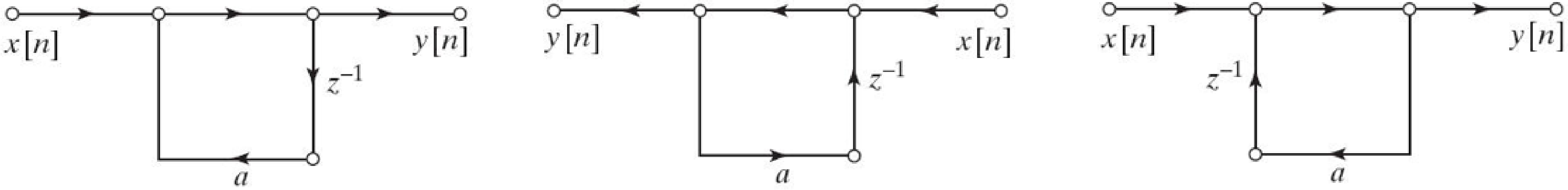




Transposed forms

- Reverse the directions of all branches in the network
- Keep functions on branches (multiplications, delays, etc) the same
- Reverse the input and output
  ➔ Obtain the same system!

- Simple example $H(z) = \dfrac{1}{1 - az^{-1}}$



(a) (b) (c)

Structures for FIR systems

- FIR system functions have only zeros (except for poles at z=0)

- The difference equation reduces to

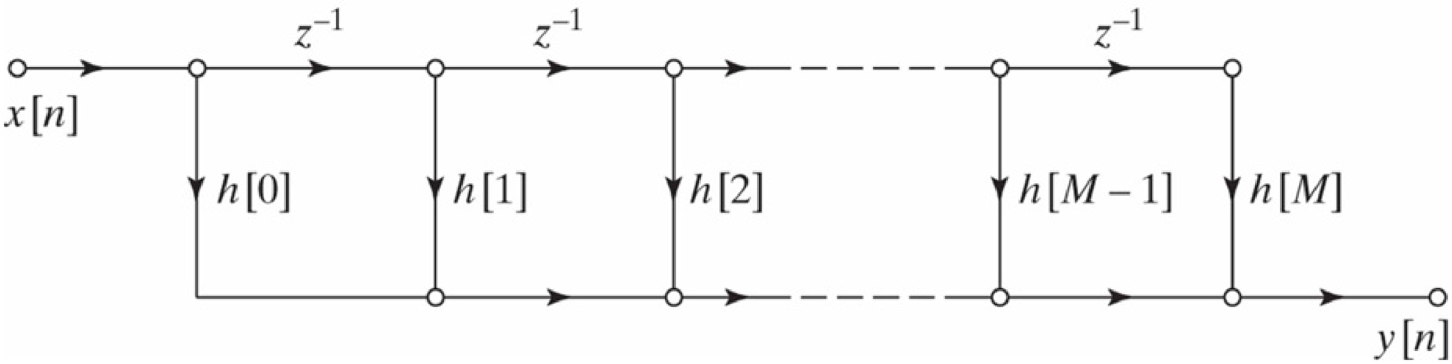
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

with the impulse response

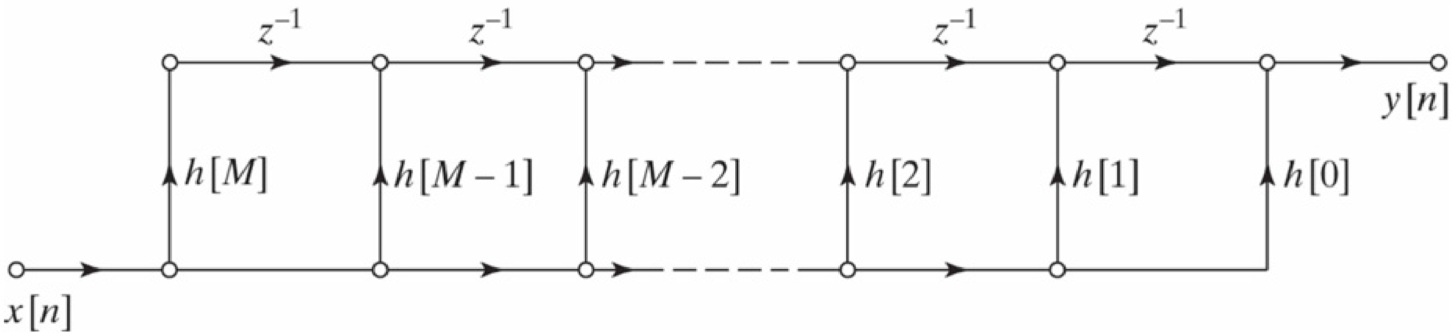
$$h[n] = \begin{cases} b_n & n = 0, 1, \ldots, M \\ 0 & \text{otherwise} \end{cases}$$

Direct form

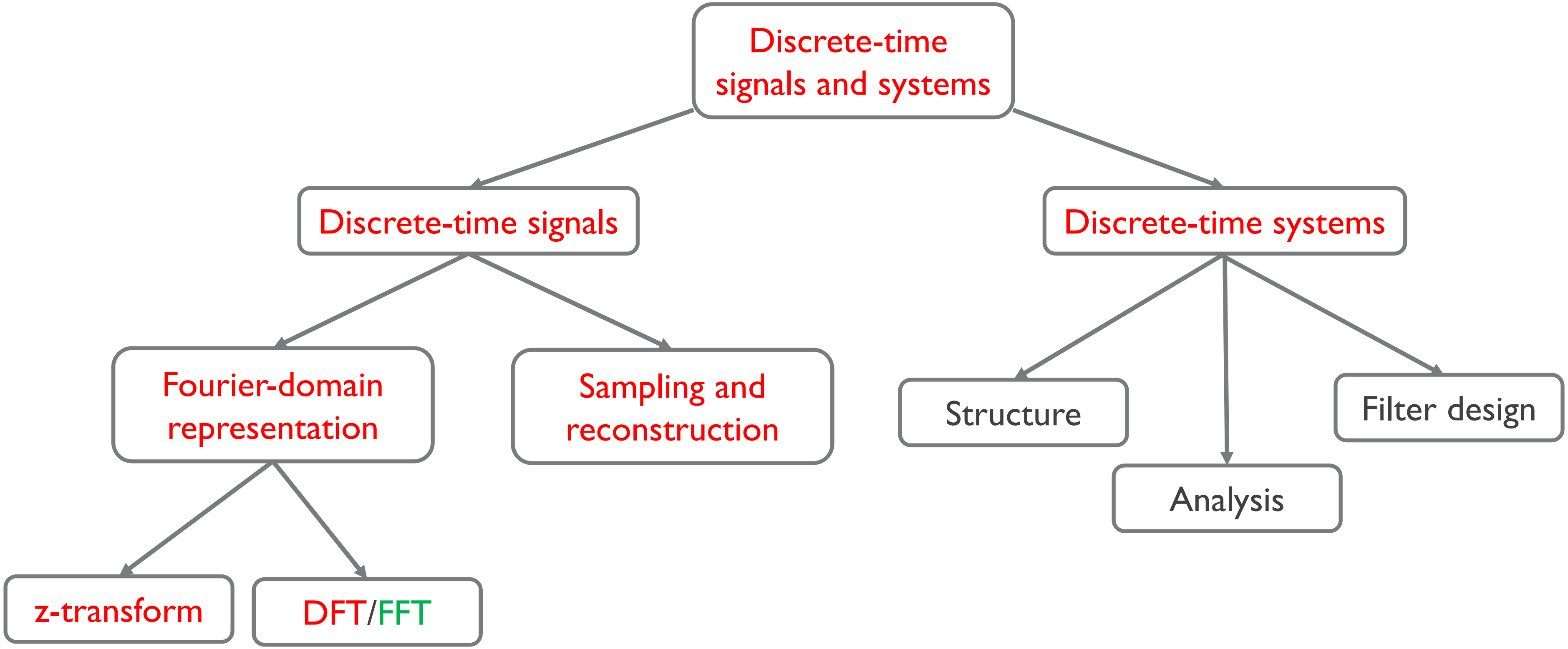
- Direct form



- Transposed direct form



Course at glance



Discrete-time signals and systems

Discrete-time signals

Discrete-time systems

Fourier-domain representation

Sampling and reconstruction

Structure

Analysis

Filter design

z-transform

DFT/FFT

Discrete Fourier Transform (DFT)

- Both time-domain sequence $x[n]$ and its DFT $X[k]$ are discrete sequences
  ➔ Appropriate for digital processing

- The complexity of direct computations

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

proportional to N^2
➔ May not be feasible for large *N*

Complexity of direct DFT

- $x[n]$ is complex sequence in general

- To compute DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} \left[(\mathrm{Re}\{x[n]\}\mathrm{Re}\{W_N^{kn}\} - \mathrm{Im}\{x[n]\}\mathrm{Im}\{W_N^{kn}\}) \right.$$
$$\left. + j(\mathrm{Re}\{x[n]\}\mathrm{Im}\{W_N^{kn}\} - \mathrm{Im}\{x[n]\}\mathrm{Re}\{W_N^{kn}\}) \right], \quad 0 \le k \le N-1$$

  - ✦ Each component of $X[k]$ requires N complex ($4N$ real) multiplications and (N-1) complex (($4N$-2) real) additions
    ➔ $X[k]$ requires N^2 complex multiplications and $N(N-1)$ complex additions

How to reduce complexity?

- Note $W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$

- Grouping

$$\mathrm{Re}\{x[n]\}\mathrm{Re}\{W_N^{kn}\} + \mathrm{Re}\{x[N-n]\}\mathrm{Re}\{W_N^{k(N-n)}\}$$

$$= (\mathrm{Re}\{x[n]\} + \mathrm{Re}\{x[N-n]\})\mathrm{Re}\{W_N^{kn}\}$$

$$- \mathrm{Im}\{x[n]\}\mathrm{Im}\{W_N^{kn}\} - \mathrm{Im}\{x[N-n]\}\mathrm{Im}\{W_N^{k(N-n)}\}$$

$$= -(\mathrm{Im}\{x[n]\} - \mathrm{Im}\{x[N-n]\})\mathrm{Im}\{W_N^{kn}\}$$

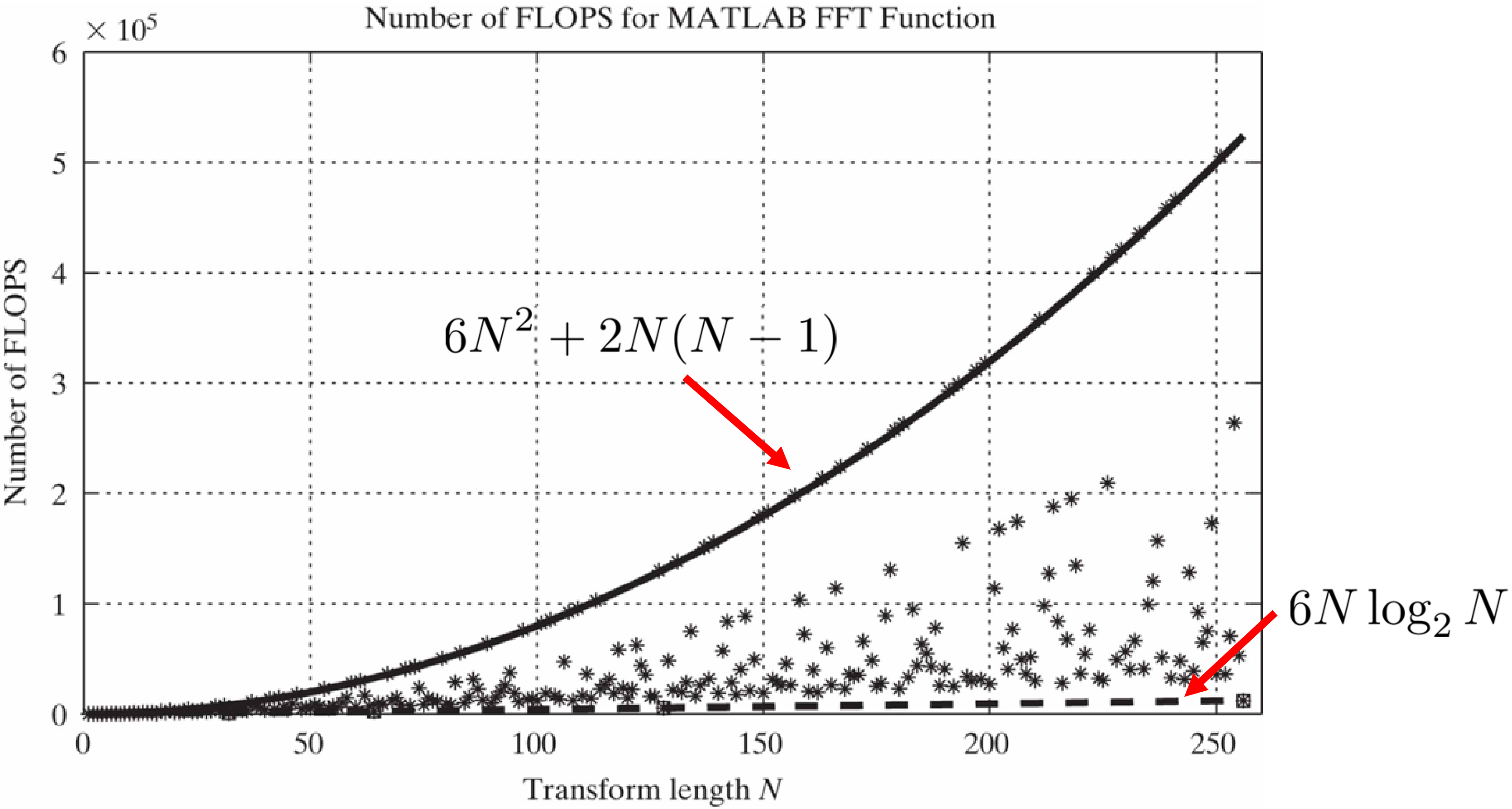
➔ Number of multiplications can be reduced by approximately a factor of 2

Fast Fourier Transform (FFT)

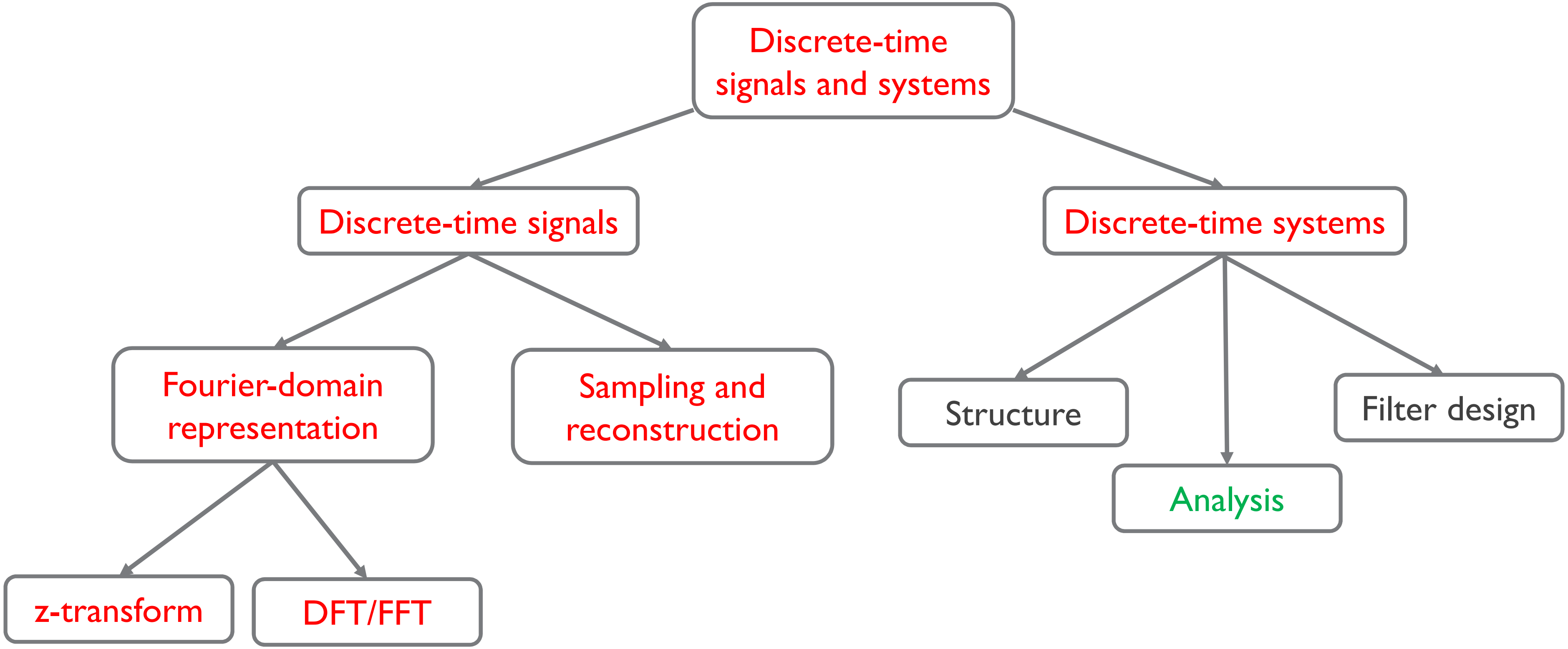
- A class of algorithms for efficient computation of DFT
- Computation proportional to $N \log_2 N$
- FFT algorithms not applicable for all values of *N*
- In general, FFT works for $N = 2^v$ for arbitrary positive integer v

Complexity comparison

Figure 9.26 Number of floating-point operations as a function of N for MATLAB fft ( ) function (revision 5.2).



Course at glance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LTI System Analysis

Review of transformations

- LTI system in time-domain

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n]h[n-k]$$

- LTI system in frequency-domain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- LTI system with z-transform

$$Y(z) = H(z)X(z) \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Magnitude and phase of output

- Fourier transform is in general a complex number

$$\begin{aligned} H(e^{j\omega}) &= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\ &= |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \end{aligned}$$

- Magnitude and phase of the Fourier transforms of system input and output

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

$$|H(e^{j\omega})| : \ \text{magnitude response or gain}$$

$$\angle H(e^{j\omega}) : \ \text{phase response or phase shift}$$

Effect on magnitude and phase

- The effects

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

  may or may not be desirable

- For undesirable effects, often refer to the effects as magnitude and phase distortions

Phase of general complex numbers

- Phase is not uniquely defined
  - Period of 2π

- Denote the principal value of the phase of $H(e^{j\omega})$

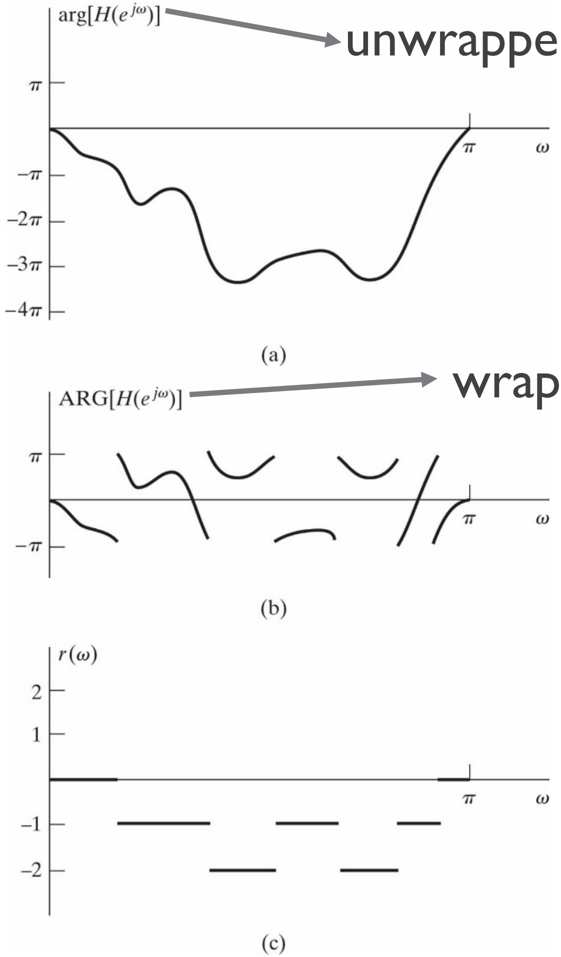
$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi$$

- General angle notation

$$\angle H(e^{j\omega}) = \arg[H(e^{j\omega})] = \text{ARG}[H(e^{j\omega})] + 2\pi r(\omega)$$

Arbitrary integer that may depend on ω

Discontinuity of principal value of phase



unwrapped continuous phase

wrapped continuous phase

Group delay

- The group delay is defined as

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

- Using other angles $\mathrm{ARG}[H(e^{j\omega})],\ \angle H(e^{j\omega})$ also possible
  - Need to take possible discontinuities into account

Example of ideal delay system

- Impulse response

$$h_{\text{id}}[n] = \delta[n - n_d]$$

- Frequency response

$$H_{\text{id}}(e^{j\omega}) = e^{-j\omega n_d}$$

- Magnitude and phase responses

$$|H_{\text{id}}(e^{j\omega})| = 1$$

$$\angle H_{\text{id}}(e^{j\omega}) = \underbrace{-\omega n_d}_{\text{Phase response linear with frequency}}, \quad |\omega| < \pi$$

Delay distortion and phase distortion

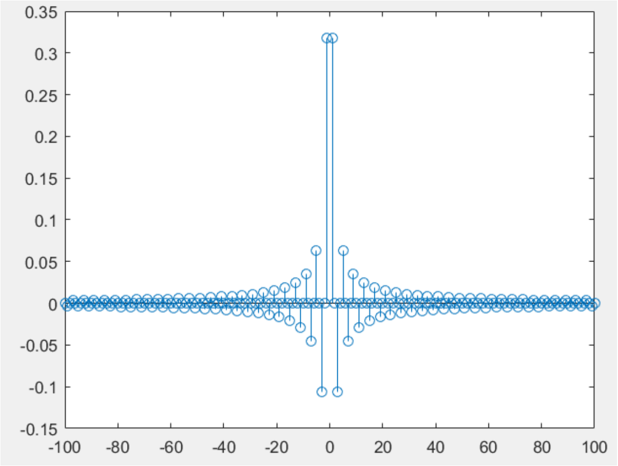
- In general, phase distortion is nonlinear with frequency
- Delay distortion (i.e., linear phase distortion) is rather a mild phase distortion
  - Can be compensated easily
- When designing filters or other LTI systems, frequently accept a linear-phase response (while zero-phase response is ideal)

Lowpass filter example

- Ideal lowpass filter with zero delay

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

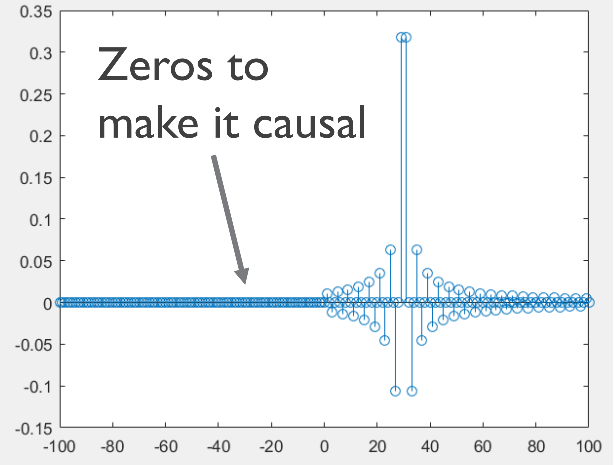
$$h_{\text{lp}}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$



- Ideal lowpass filter with delay

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$h_{\text{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -\infty < n < \infty$$



Zeros to make it causal

Narrowband input

- Assume $X(e^{j\omega})$ is a narrowband signal, i.e., $x[n] = s[n]\cos(\omega_0 n)$
  - $X(e^{j\omega})$ is nonzero only around $\omega = \omega_0$

- The effect of the phase of the system can be linearly approximated as

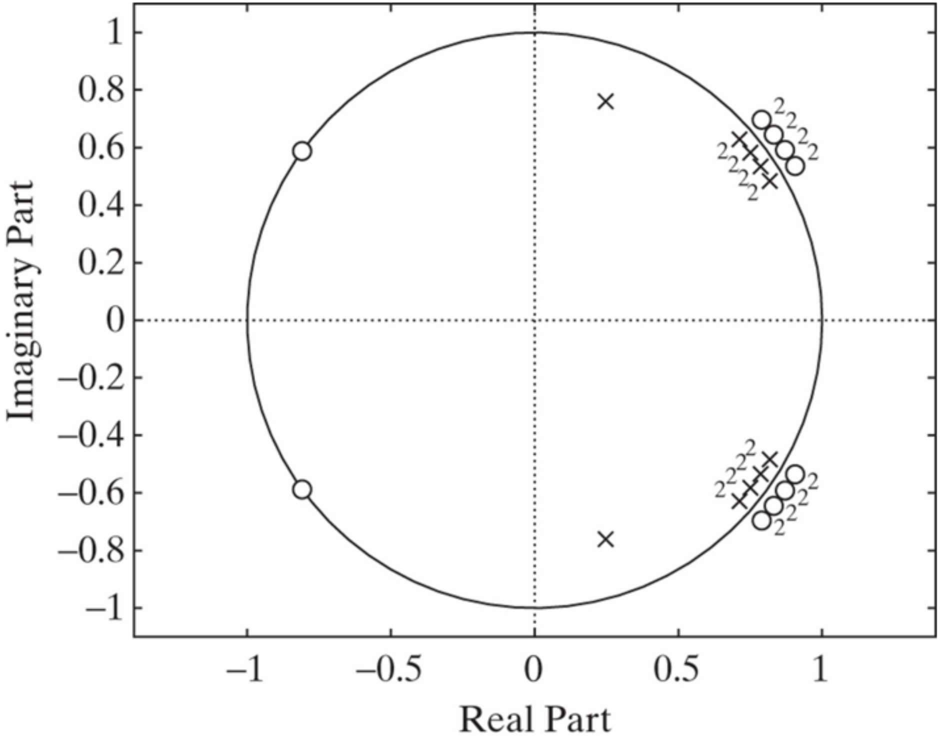
$$\arg[H(e^{j\omega})] \simeq \underbrace{-\phi_0}_{\text{Constant term}} - \underbrace{\omega n_d}_{\text{Group delay}}$$

- Consider wideband signal as a superposition of narrowband signals
  - Constant group delay with frequency ➔ each narrowband component will undergo identical delay
  - Nonconstant group delay ➔ result in time dispersion

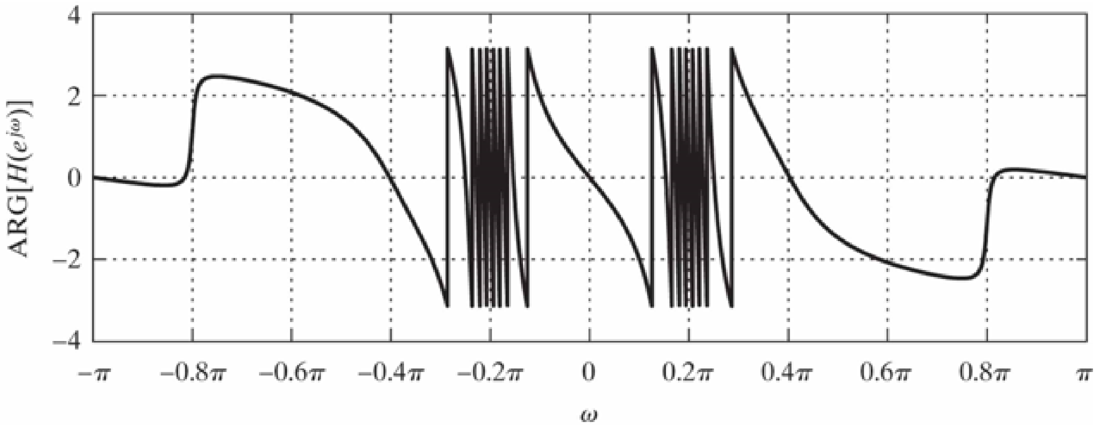
Effects of group delay and attenuation

- Consider the system with pole-zero plot

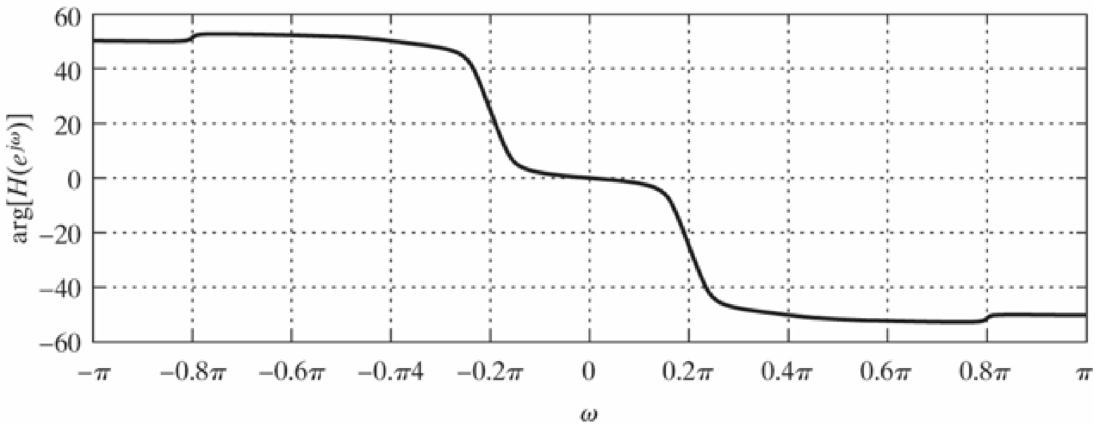
Figure 5.2 Pole-zero plot for the filter in the example of Section 5.1.2. (The number 2 indicates double-order poles and zeroes.)



Phase response

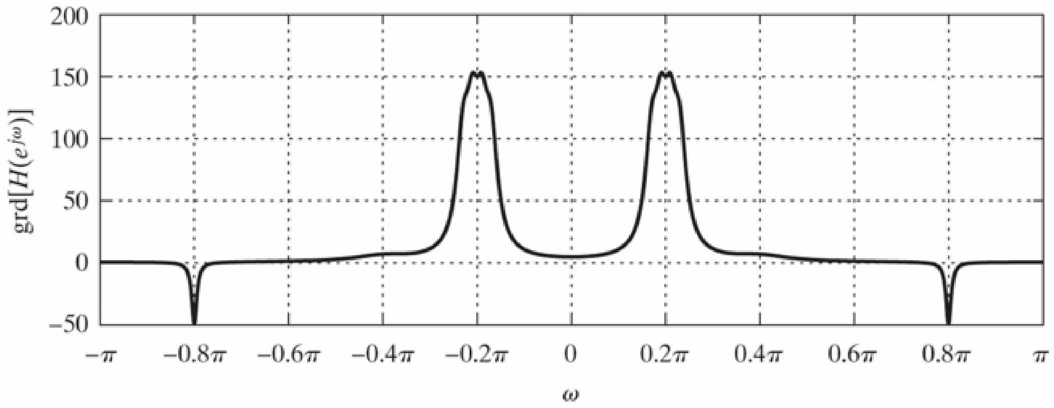


(a) Principle Value of Phase Response

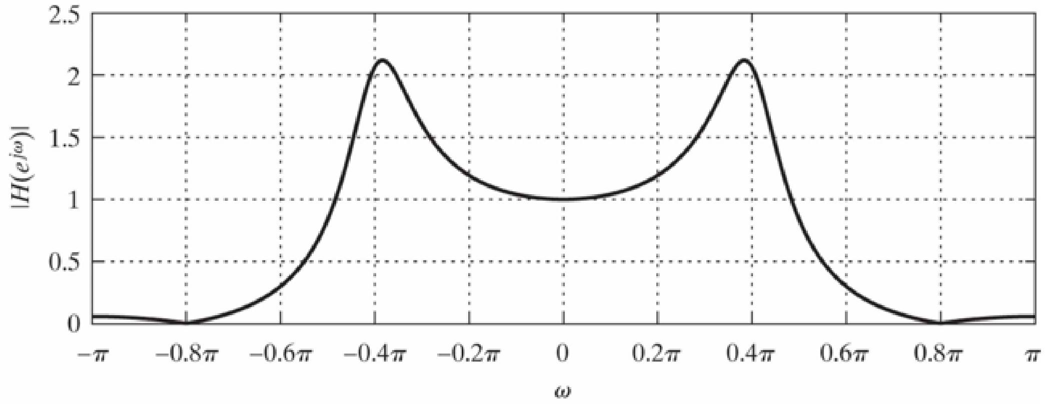


(b) Unwrapped Phase Response

Group delay and magnitude response

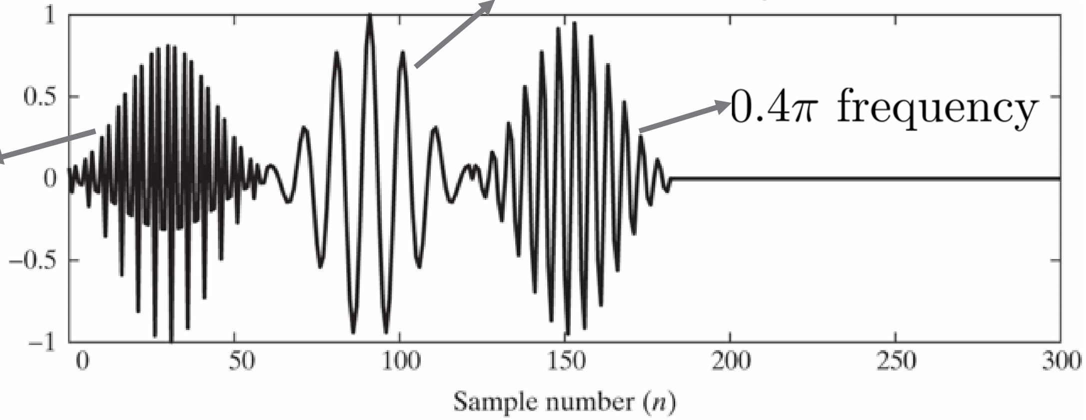


(a) Group delay of $H(z)$



(b) Magnitude of Frequency Response

Input sequence

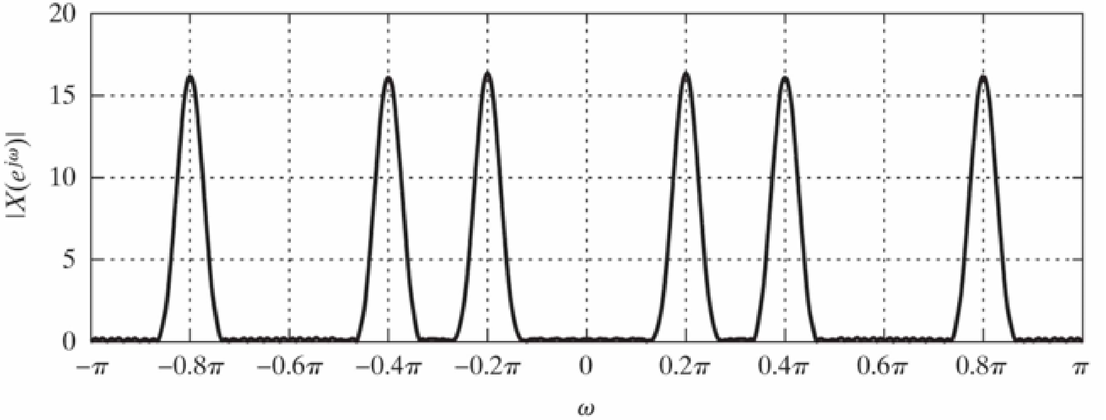


0.2π frequency

0.4π frequency

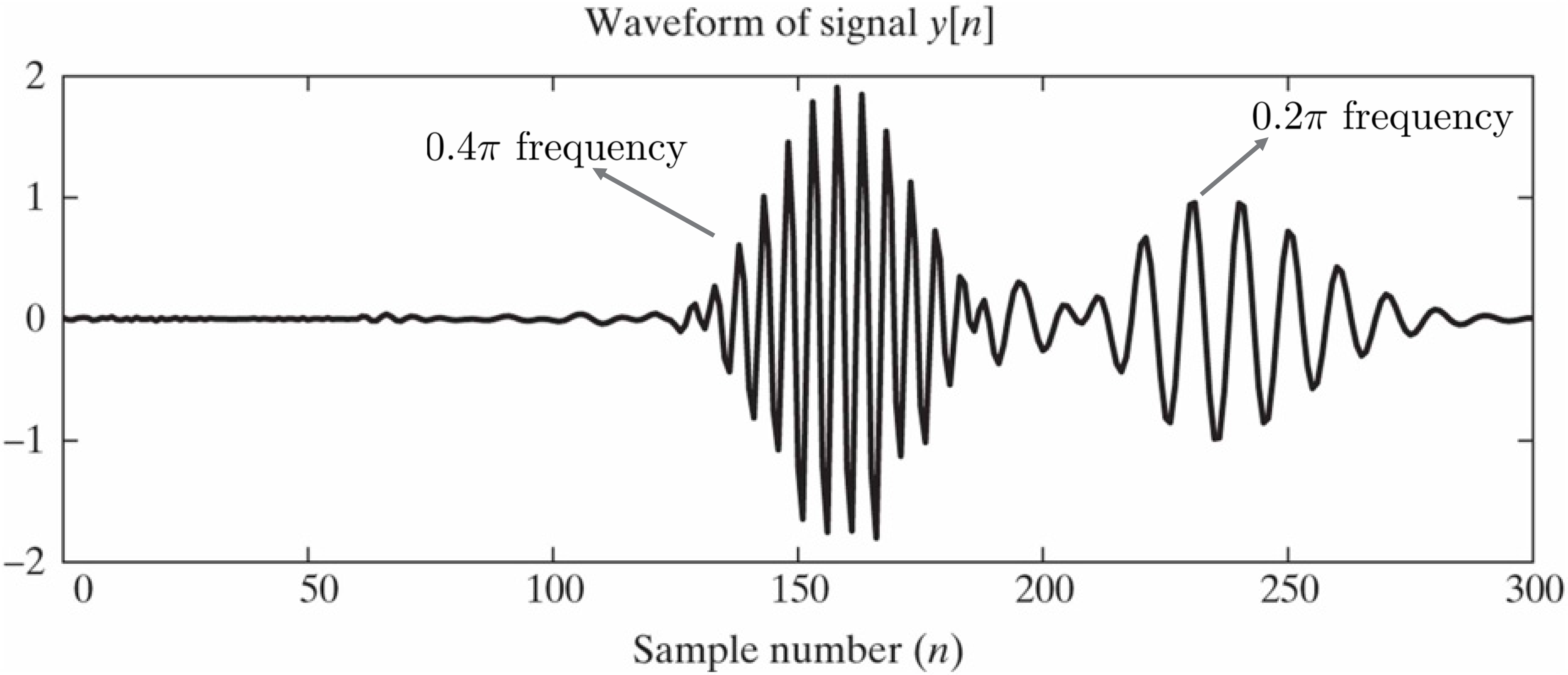
0.8π frequency

(a) Waveform of signal $x[n]$



(b) Magnitude of DTFT of $x[n]$

Output sequence



Review from Previous Lectures

z-transform for difference equations

- z-transform is particularly useful for LTE systems with difference equations

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

- Due to linearity and time-shift properties

$$Y(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z) + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z)$$

$$\Rightarrow Y(z) = \left(\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}\right)X(z)$$

$H(z)$!

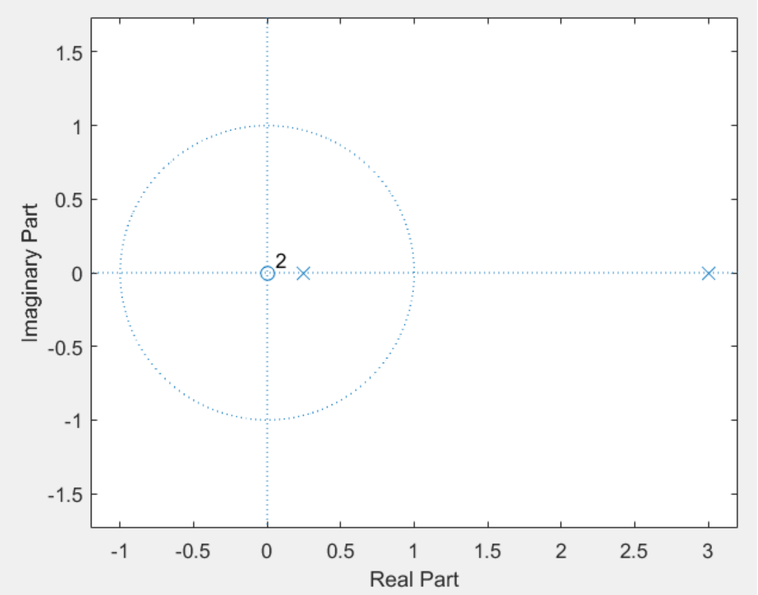
Example

- Let $y[n] = \frac{13}{4}y[n-1] - \frac{3}{4}y[n-2] + x[n]$
- z-transform gives

$$Y(z) = \frac{13}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{13}{4}z^{-1} + \frac{3}{4}z^{-2}}$$

$$= \frac{z^2}{z^2 - \frac{13}{4}z + \frac{3}{4}} = \frac{z^2}{\left(z - \frac{1}{4}\right)(z-3)}$$



Example

- Using partial fraction expansion

$$H(z) = -\frac{1}{11}\frac{z}{z-\frac{1}{4}} + \frac{12}{11}\frac{z}{z-3}$$

- Three possibilities for ROC
  - $|z| < \frac{1}{4}$
  - $\frac{1}{4} < |z| < 3$
  - $|z| > 3$

Example

- If $|z| < \frac{1}{4}$

- Impulse response becomes

$$h[n] = \frac{1}{11}\left(\frac{1}{4}\right)^n u[-n-1] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Left-sided sequence

- BIBO stable? No! $\lim_{n \to -\infty} |h[n]| = \infty$

Example

- If $\frac{1}{4} < |z| < 3$

- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Two-sided sequence

- BIBO stable? Yes! $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Example

- If $|z| > 3$

- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{11}(3)^n u[n]$$

- Causal? Yes!

- BIBO stable? No!

Stability and causality

- Stability requires ROC to include unit circle $|z| = 1$
  - Proof using triangle inequality $|a + b| \le |a| + |b|$

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}| \overset{|z|=1}{=} \underbrace{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{For BIBO stability}}$$

- Causality requires ROC to satisfy $|z| > |p_N|$

  Largest pole

- If the system to be stable AND causal
  ➔ $|p_N| < 1$
  ➔ All poles must be located within unit circle

New Concepts

Inverse systems

- Definition: $H_i(z) = \dfrac{1}{H(z)}$
- Time-domain condition: $h[n] * h_i[n] = \delta[n]$
- Frequency response of inverse system (if it exists): $H_i(e^{j\omega}) = \dfrac{1}{H(e^{j\omega})}$
  - Not all systems have an inverse
    - Ideal lowpass filter does not have an inverse

Inverse with rational form

- Consider

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

  - ✦ Zeros at $z = c_k$ and poles at $z = d_k$ with possible zeros and/or poles at $z = 0$ and $z = \infty$

- Inverse $H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$

➔ The poles (zeros) of $H_i(z)$ are the zeros (poles) of $H(z)$

Pole/zeros relation

- The time-domain condition $h[n] * h_i[n] = \delta[n]$ states ROCs of $H(z)$ and $H_i(z)$ should overlap
- For causal $H(z)$, ROC is $|z| > \max_k |d_k|$
- Any appropriate ROC for $H_i(z)$ that overlaps with $|z| > \max_k |d_k|$ is a valid ROC for $H_i(z)$

Example I

- Let $H(z) = \dfrac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$ with ROC $|z| > 0.9$
- The inverse becomes $H_i(z) = \dfrac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$
  - Two possible ROCs
  - Only $|z| > 0.5$ overlaps with $|z| > 0.9$
- Impulse response with proper ROC becomes

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

  - The inverse system both causal and stable

Example 2

- Let $H(z) = \dfrac{z^{-1} - 0.5}{1 - 0.9z^{-1}}$ with ROC $|z| > 0.9$
- The inverse becomes $H_i(z) = \dfrac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \dfrac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$
  - Two possible ROCs
  - Both regions overlap with $|z| > 0.9$
- Possible impulse responses

$$h_{i1}[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n] \text{ with ROC } |z| < 2$$

➡ Stable, noncausal

$$h_{i2}[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1] \text{ with ROC } |z| > 2$$

➡ Unstable, causal

Generalization of inverse system

- If causal $H(z)$ has zeros at $c_k, k = 1, \ldots, M$, its inverse will be causal iff the ROC is

$$|z| > \max_k |c_k|$$

- If the inverse $H_i(z)$ to be stable, the ROC of $H_i(z)$ must include unit circle

$$\max_k |c_k| < 1$$

➔ All the zeros of $H(z)$ must be inside unit circle

- If both poles and zeros of $H(z)$ are inside unit circle
➔ Both $H(z)$ and its inverse $H_i(z)$ are causal and stable
➔ Referred to as minimum-phase systems (will be discussed shortly)

Homework

- Problems in textbook: 5.29, 5.46, 5.48

- MATLAB problem
  - Write a program to plot Figs. 5.2-5.6 in the textbook Section 5.1.2.
  - Due: 12/11 (Tuesday)
  - Send one m file (with proper annotations) that plots all figures at once
  - **Please use the same step sizes for x- and y-axes!!!**
    - **Otherwise, -0.5 point per plot**
    - A gain scaler is missing in the equation (5.15).
      - In Fig. 5.4(b), |H(e^{jw}|=1 when w=0 while the equation (5.15) does not give this result.
      - Please have a proper scaler to have the same figure as in Fig. 5.4(b).