Digital Signal Processing

POSTECH

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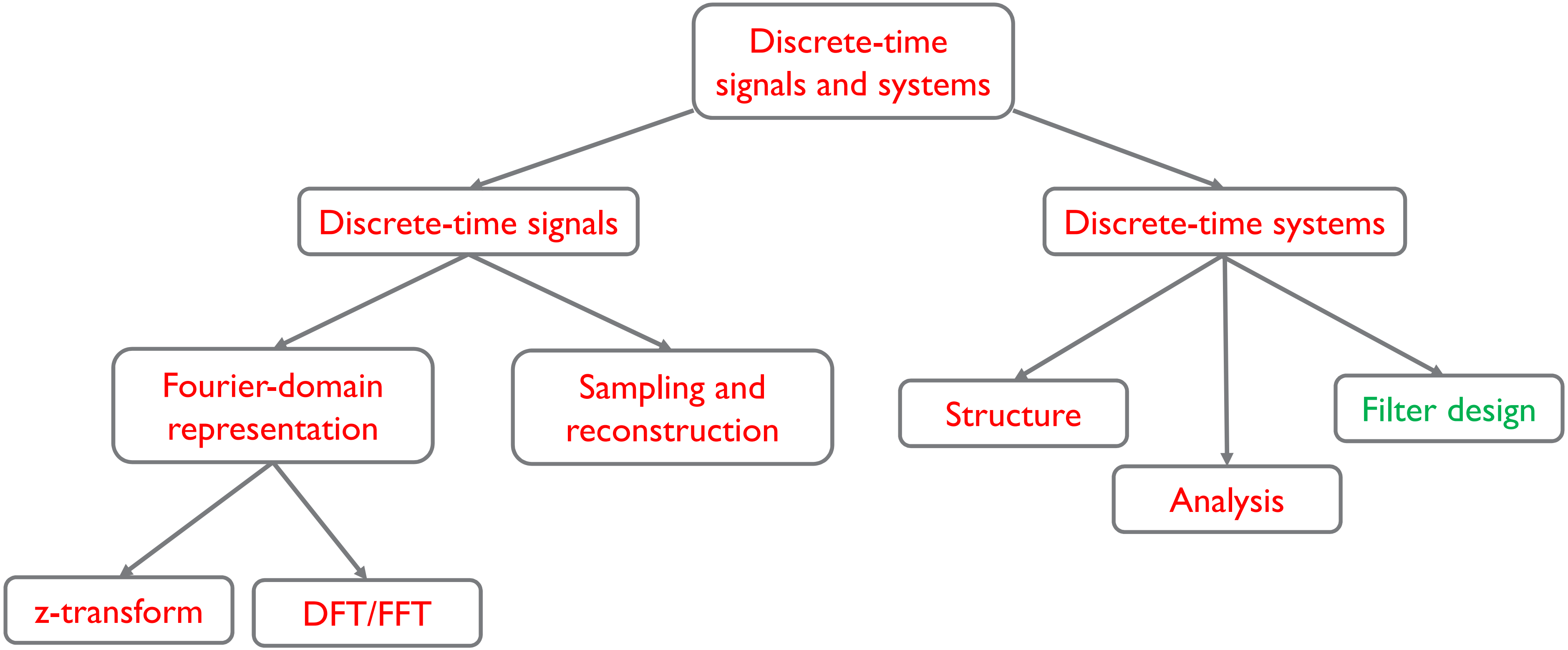
Course at glance



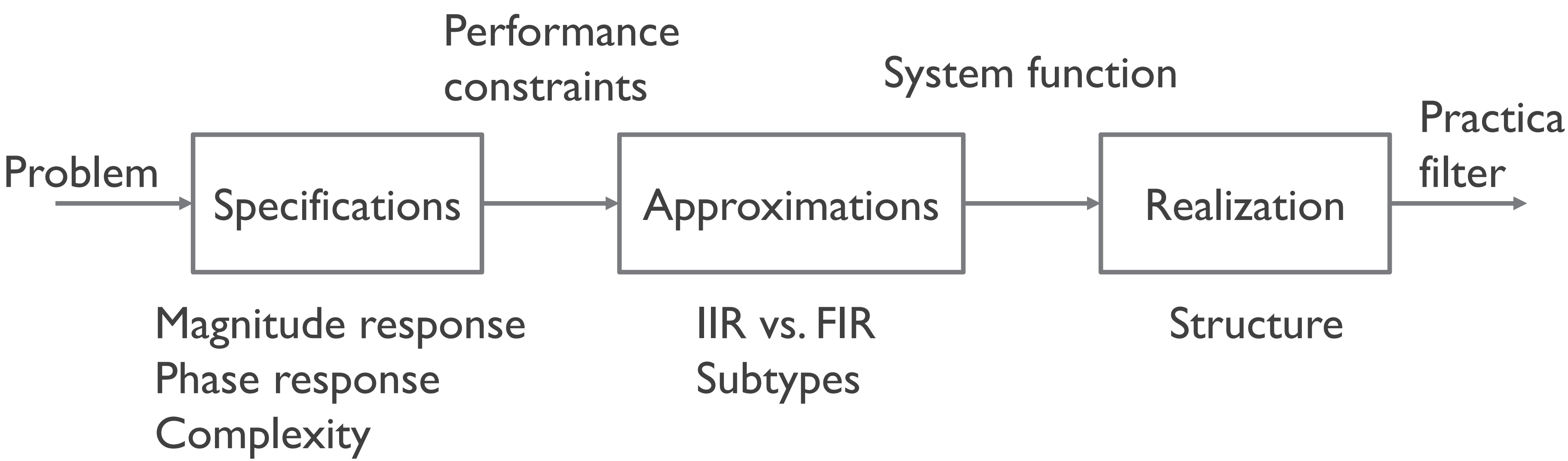
Definition of filter

- Filter, in broader sense, covers any system
  - Distortion environments are also filters

- We denote filters as controllable systems here

Filter design process

- Three design steps



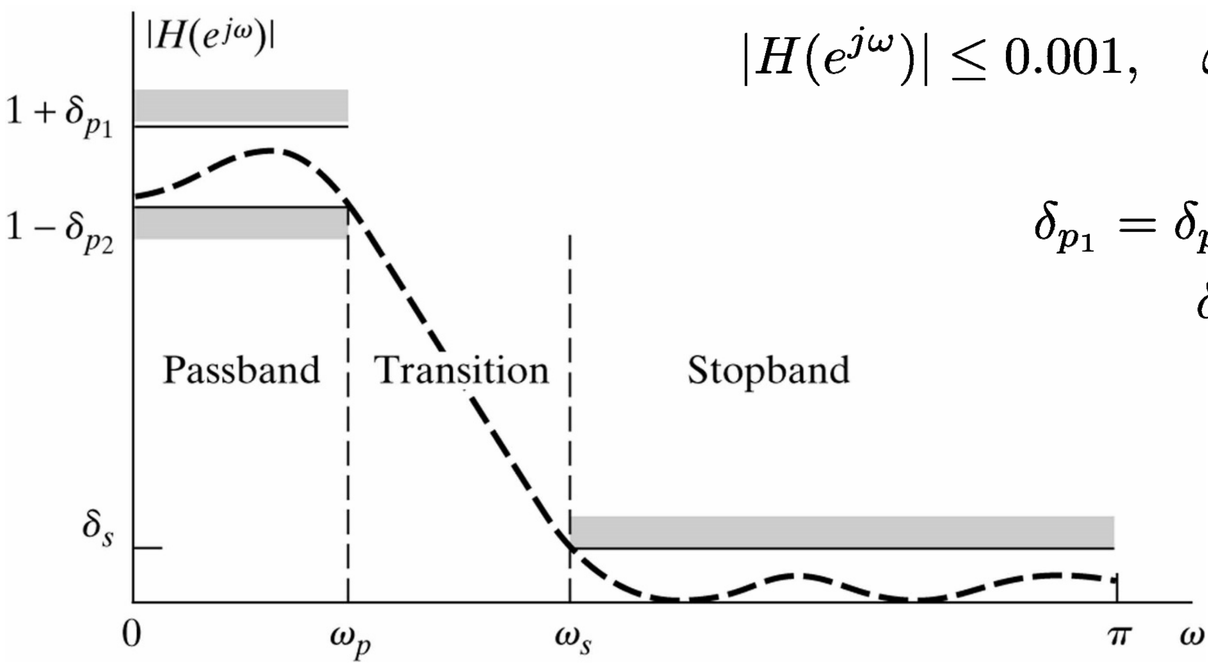
- Focus on lowpass filters
  - Can be generalized to other frequency-selective filters

Example specifications

- Specifications for a discrete-time lowpass filter

$$1 - 0.01 \le |H(e^{j\omega})| \le 1 + 0.01, \quad 0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le 0.001, \quad \omega \ge \omega_s$$



$$\delta_{p_1} = \delta_{p_2} = 0.01$$

$$\delta_s = 0.001$$

Specifications of frequency response

- Typical lowpass filter specifications in terms of tolerable
  - Passband distortion → as **smallest** as possible
  - Stopband attenuation → as **greatest** as possible
  - Width of transition band → as **narrowest** as possible

- Improving one often worsens others → tradeoff exists

- Increasing filter order may improve all → increase complexity

Design a filter

- Design goal
  ➔ Find system function to make frequency response meet the specifications (tolerances)

- Infinite impulse response (IIR) filter
  - Poles insider unit circle due to causality and stability
  - Rational function approximation

- Finite impulse response (FIR) filter
  - For filters with linear phase requirement
  - Polynomial approximation

Example of IIR filter design

- For rational (and stable and causal) system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

find the system coefficients such that the corresponding frequency response

$$H(e^{j\omega}) = H(z)\,|_{z=e^{j\omega}}$$

provides a good approximation to a desired response

$$H(e^{j\omega}) \approx H_{\text{desired}}(e^{j\omega})$$

IIR vs. FIR

- Either FIR or IIR is often dependent on the phase requirements
- Only FIR filter can be at the same time stable, causal and GLP
- Design principle
  - If GLP is required → FIR
  - If not → IIR preferable because IIR can meet specifications with lower complexity

IIR vs. FIR

- IIR
  - Rational system function
  - Poles and zeros
  - Stable/unstable
  - Hard to control phase
  - Low order (4-20)
  - Designed on the basis of analog filter
- FIR
  - Polynomial system function
  - Only zeros
  - Always stable
  - Easy to get (generalized) linear phase
  - High order (20-200)
  - Usually unrelated to analog filter designs

IIR Filter Design

Discrete-time IIR filters from continuous-time filters

- Continuous-time (or analog) IIR filter design is highly advanced
  - Relatively simple closed-form design possible

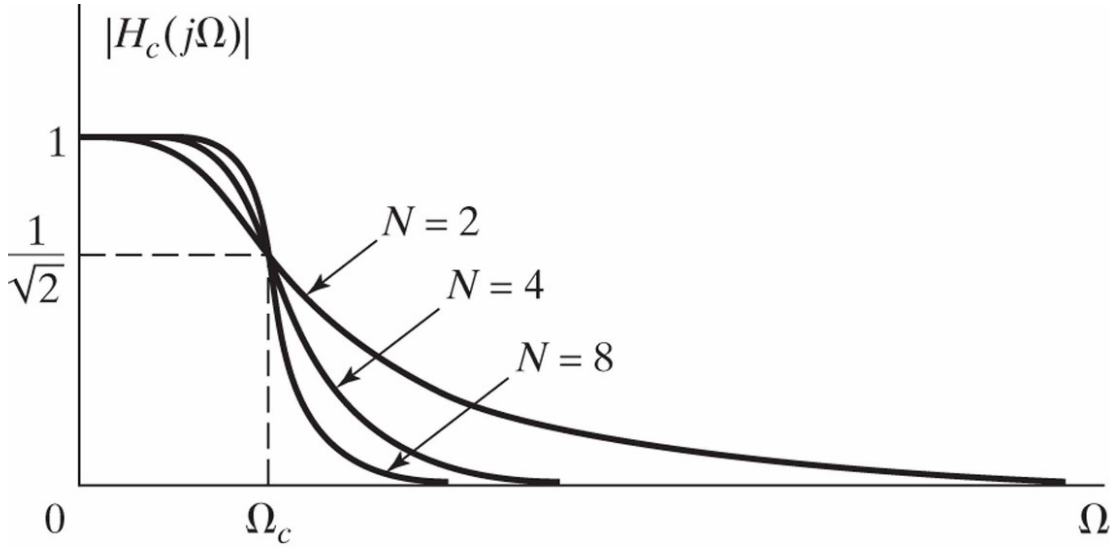
- Discrete-time IIR filter design
  - Filter specifications for discrete-time filter
  - Convert to continuous-time specifications
  - Design continuous-time filter
  - Convert to discrete-time filter
    - Impulse invariance method
    - Bilinear transformation method

Analog filter designs

- Butterworth filter
- Type I Chebyshev filter
- Type II Chebyshev filter
- Elliptic filter

Butterworth lowpass filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$
  - Two parameters
    - Order N
    - Cutoff frequency Ω_c
  - Monotonic in both passband and stopband

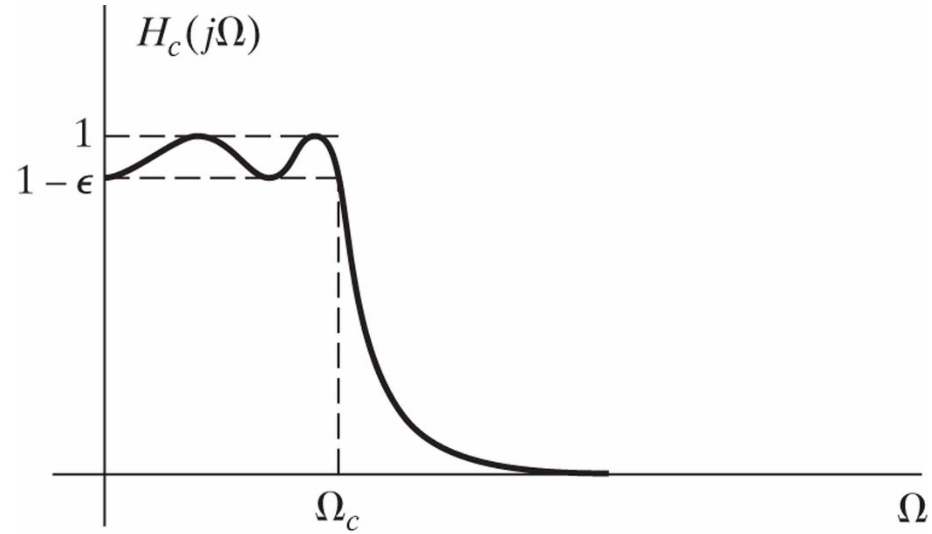


Type I Chebyshev lowpass filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 V_N^2(\Omega/\Omega_c)}$

  where $V_N(x) = \cos(N\cos^{-1} x)$
  - Three parameters
    - Order N
    - Cutoff frequency Ω_c
    - Allowable passband ripple ϵ

- $|H_c(j\Omega)|^2$ has equi-ripple error in passband and monotonic in stopband



Type II Chebyshev lowpass filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1 + [\epsilon^2 V_N^2(\Omega/\Omega_c)]^{-1}}$

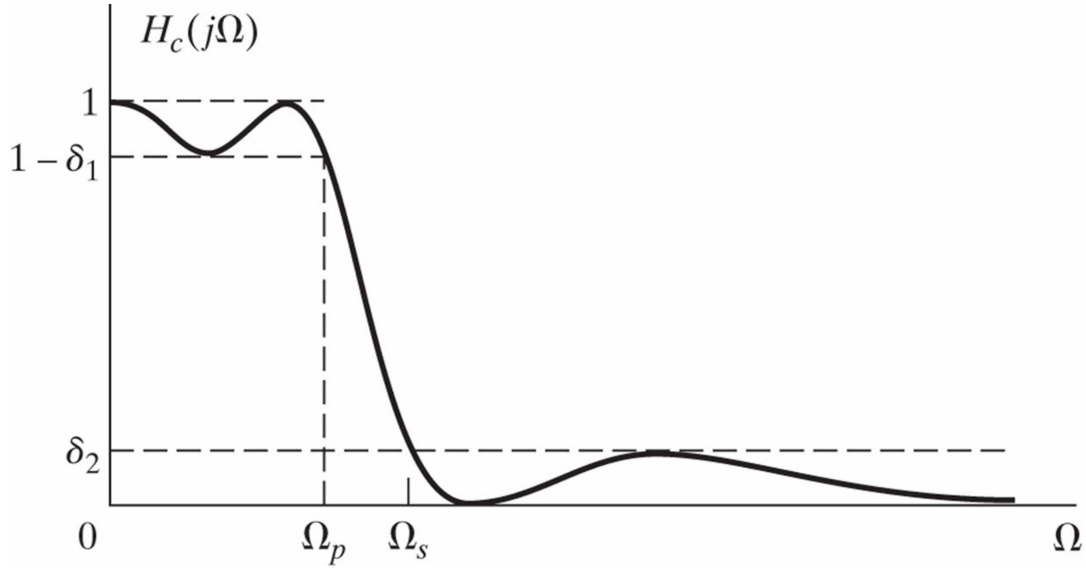
- Similar to Type I Chebyshev lowpass filter
  - $|H_c(j\Omega)|^2$ now has equi-ripple error in stopband and flat in passband

Elliptic filter

- Filter form $|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 U_N^2(\Omega)}$

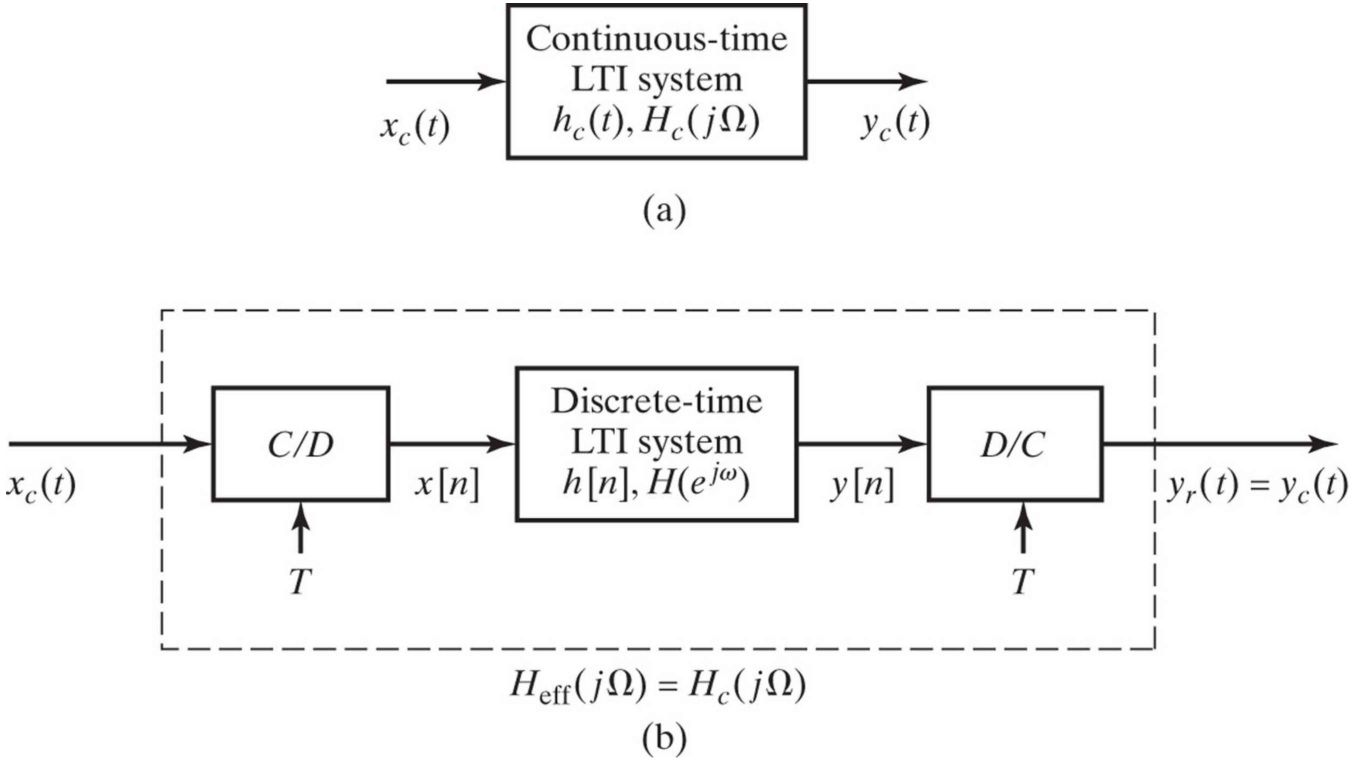
  where $U_N(\Omega)$ is a Jacobian elliptic function

- $|H_c(j\Omega)|^2$ has equi-ripples in both passband and stopband



Discrete-time IIR filter design – impulse invariance

- Recall "discrete-time processing of continuous-time signals" in Section 4.4



Output signal

- Necessary conditions
  - The discrete-time system is LTI
  - Continuous-time signal $x_c(t)$ is bandlimited
  - Sampling rate Ω_s is at or above the Nyquist rate $2\Omega_N$

- If all conditions are satisfied, the output signal becomes

$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega)$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Cutoff frequency of ideal lowpass filter

$$\omega = \Omega T$$

Impulse invariance

- Recall $H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$

- We want to have $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

$$H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi$$

- In time-domain: $h[n] = Th_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$H(e^{j\omega}) = T\frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

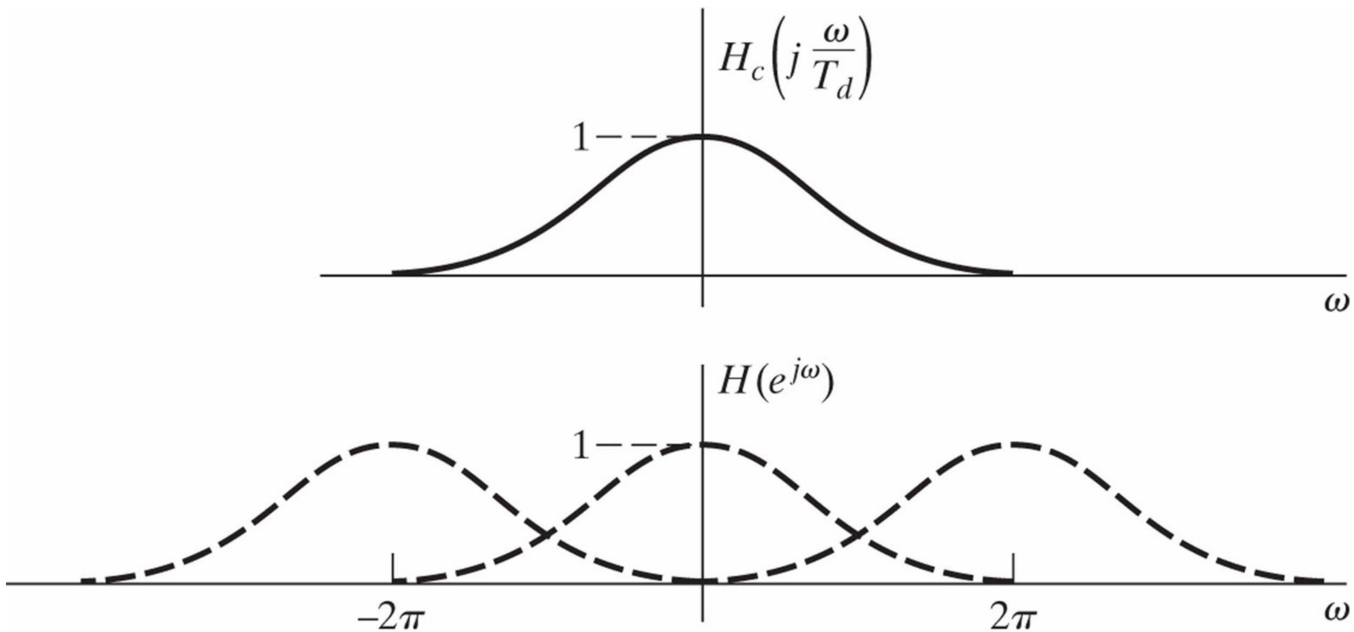
$$= H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Because $H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$

Only true when the filter is bandlimited

Impulse invariance - aliasing

- If the analog filter is not bandlimited (typically the case in practice)
  ➔ Aliasing occurs in the discrete-time filter
  - Impulse invariance not appropriate for designing highpass filters



How can we avoid the aliasing?

- Consider higher sampling frequency for analog filter $\Omega_s = 1/T$
- Will this work? No!
  - Filter specifications given from discrete-time filter requirements
  - The specifications transformed to continuous-time by $\Omega = \omega/T$
  - Continuous-time filter designed by continuous-time specifications
  - Final discrete-time filter obtained by impulse invariance method (sampling)

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

→ Effect of $\Omega_s = 1/T$ cancels out

- Aliasing can be avoided by overdesigning analog filter

Interpretation using system functions

- Transformation from continuous-time system to discrete-time system is easy to carry out using system functions
- After partial fraction expansion

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} h[n] &= T_d h_c(nT_d) \\ &= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \\ &= \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n] \end{aligned}$$

$$\stackrel{\mathcal{Z}}{\longleftrightarrow} H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Interpretation using system functions

- Mapping from $H_c(s)$ to $H(z)$
  - Pole of $H_c(s)$ at $s = s_k$ maps to pole of $H(z)$ at $z = e^{s_k T_d}$
    ➔Stability and causality preserved
  - Continuous-time: $\mathrm{Re}\{s_k\} < 0$
  - Discrete-time: $z = e^{s_k T_d}$ inside the unit circle

Impulse invariance with Butterworth filter

- Specifications: $0.89125 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi$

$$|H(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi$$

- Since the effect $\Omega_s = 1/T$ cancels out, set *T*=1 and $\omega = \Omega$

- Transformed analog specifications

$$0.89125 \leq |H_c(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq 0.2\pi$$

$$|H_c(j\Omega)| \leq 0.17783, \quad 0.3\pi \leq |\Omega| \leq \pi$$

- Due to monotonicity of Butterworth filter

$$|H_c(j0.2\pi)| \geq 0.89125$$

$$|H_c(j0.3\pi)| \leq 017783$$

Impulse invariance with Butterworth filter

- The magnitude-squared function of Butterworth filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- From the specifications $|H_c(j0.2\pi)| \geq 0.89125, \quad |H_c(j0.3\pi)| \leq 017783$

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2, \quad 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

  - Simultaneous solutions are $N = 5.8858, \ \Omega_c = 0.70474$

    Should be integer

- Let $N = 6$ and $\Omega_c = 0.7032$ to exactly meet the passband specifications
  - Stopband specification exceeded ➔ margin for aliasing

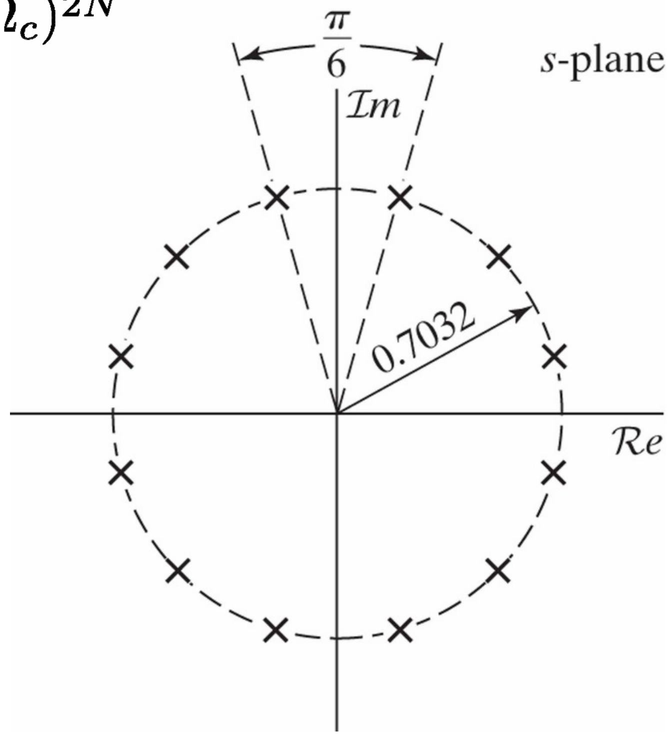
Impulse invariance with Butterworth filter

- Rewrite the magnitude-squared function

$$H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}$$

  - The system function has 12 poles

- To have a stable filter, $H_c(s)$ should have three pole pairs in the left half of s-plane



Impulse invariance with Butterworth filter

- With three pole pairs

$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

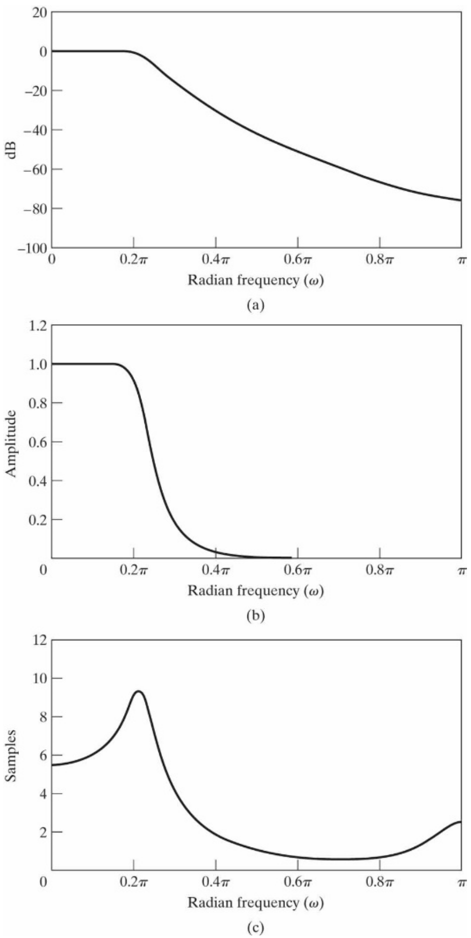
- After partial fraction, use the transformation

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

- Final discrete-time filter

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$

Impulse invariance with Butterworth filter



Discrete-time IIR filter design – bilinear transformation

- Continuous-time (analog) filter designed using s-plane (Laplace transform)

$$s = \sigma + j\Omega$$

$$H_c(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$H_c(j\Omega) = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt$$

$$z = re^{-j\omega}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

- Mapping between s-plane and z-plane

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

Rational behind bilinear transformation

- Recall $H_c(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

$$z = e^{sT}$$

T: numerical integration step size of the trapezoidal rule

Series based on area hyperbolic tangent function

$$s = \frac{1}{T}\ln(z)$$

$$= \frac{2}{T}\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right]$$

$$\approx \frac{2}{T}\frac{z-1}{z+1}$$

$$= \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Bilinear transformation - concept

- Given $s = \sigma + j\Omega$

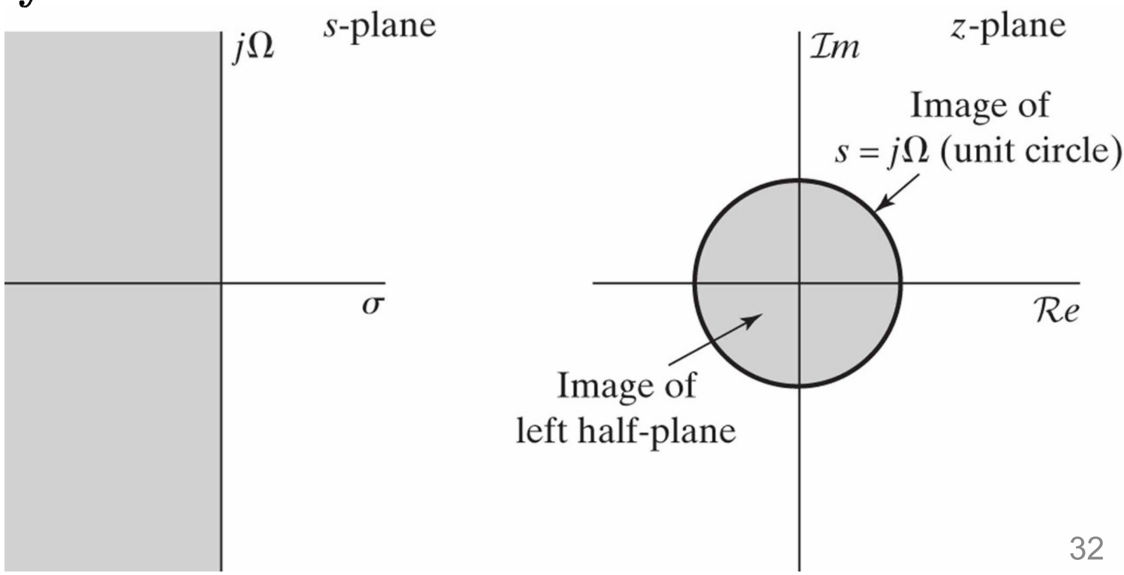
$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s} = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2}$$

  - If $\sigma < 0, \quad |z| < 1 \text{ for any } \Omega$
  - If $\sigma > 0, \quad |z| > 1 \text{ for any } \Omega$

- Given $s = j\Omega$

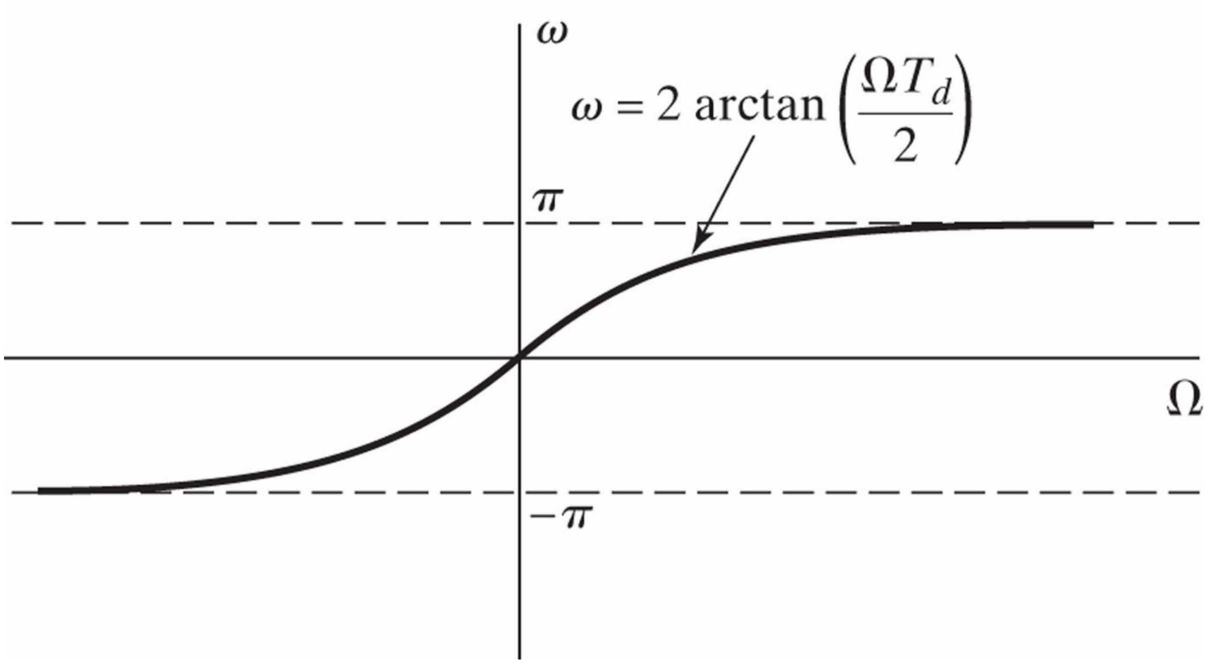
$$z = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2}$$

➔ |z|=1 for any s



Bilinear transformation – frequency relationship

- $\Omega = \dfrac{2}{T_d}\tan(\omega/2), \quad \omega = 2\arctan(\Omega T_d/2)$

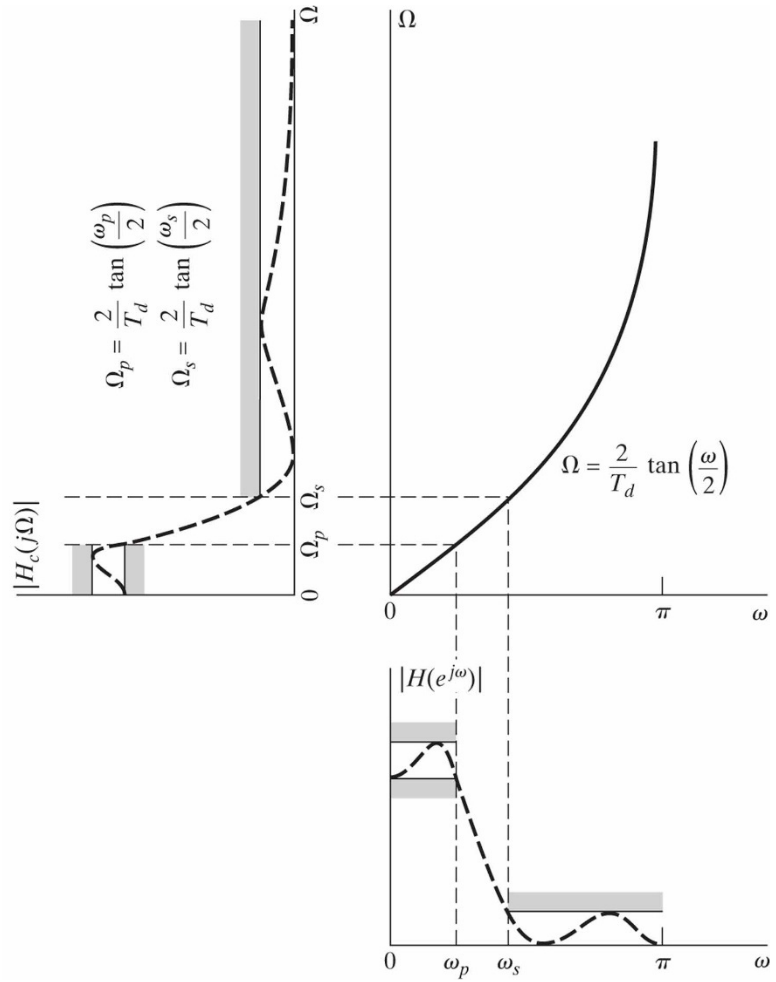


➡ Frequency warping

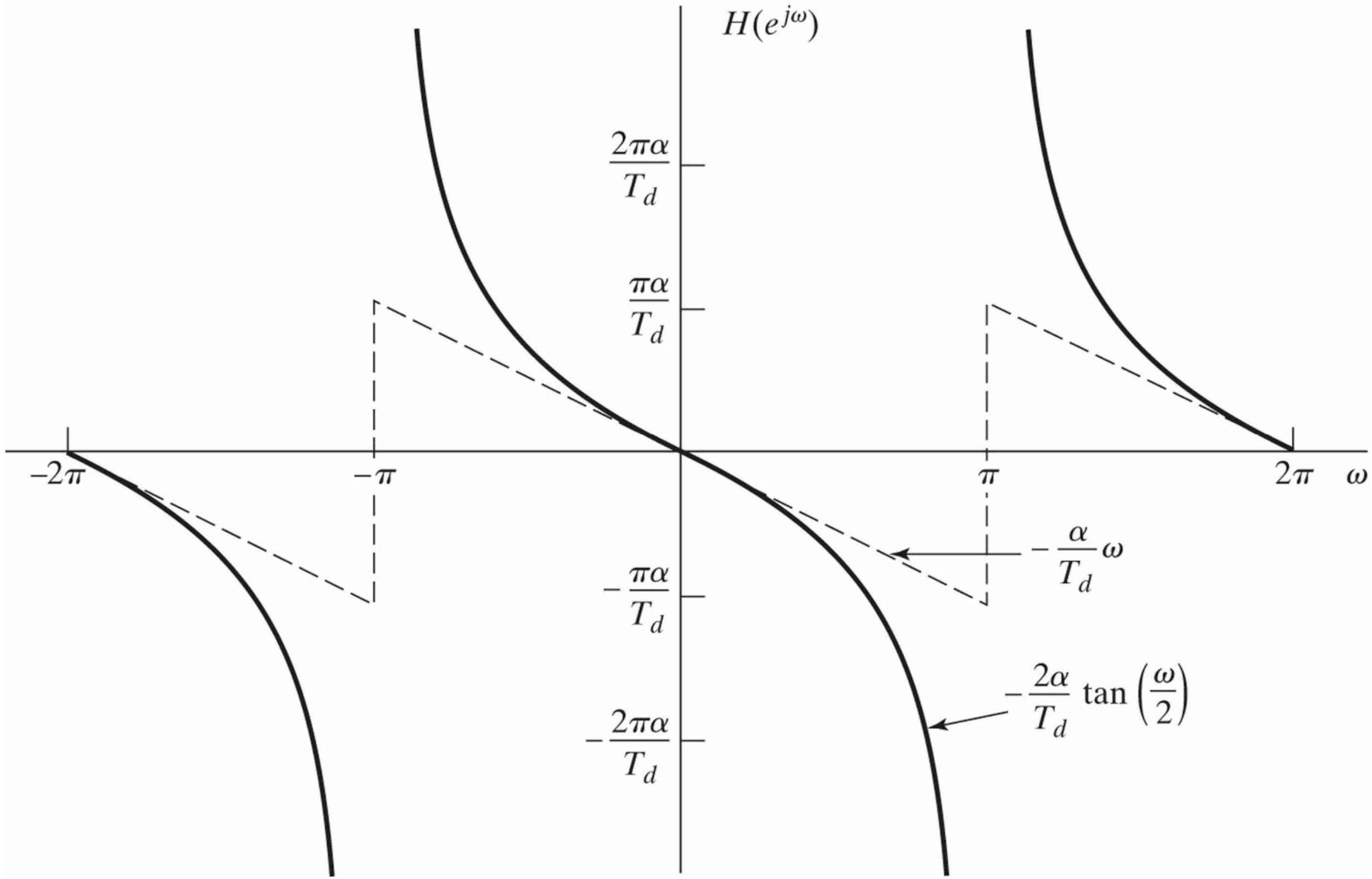
Bilinear transformation

- No problem of aliasing compared to impulse invariance method
  - Good for highpass filter design

- There exists the nonlinear compression of the frequency axis
  - Suitable for piecewise-constant magnitude response filters
  - Linear phase analog filters may lose linear phase property after transformation



Effect on phase response



Impulse invariance vs. bilinear transformation

- Bilinear transformation
  - No aliasing effect
  - Not good for preserving phase response

- Impulse invariance
  - Aliasing happens due to sampling
  - Possible to preserve linear phase of analog filter
    - Suitable to differentiator that requires linear phase

Bilinear transformation with Butterworth filter

- Specifications: $0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$

$$|H(e^{j\omega})| \le 0.17783, \quad 0.3\pi \le |\omega| \le \pi$$

- Transformed analog specifications

$$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_c(j\Omega)| \le 0.17783, \quad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| \le \infty$$

- Due to monotonicity of Butterworth filter

$$|H_c(j2\tan(0.1\pi))| \ge 0.89125, \quad |H_c(j2\tan(0.15\pi))| \le 017783$$

Bilinear transformation with Butterworth filter

- Using similar approach as in impulse invariance method, we get *N*=5.305

- Let $N = 6,\ \Omega_c = 0.766$, which now satisfies the stopband specification

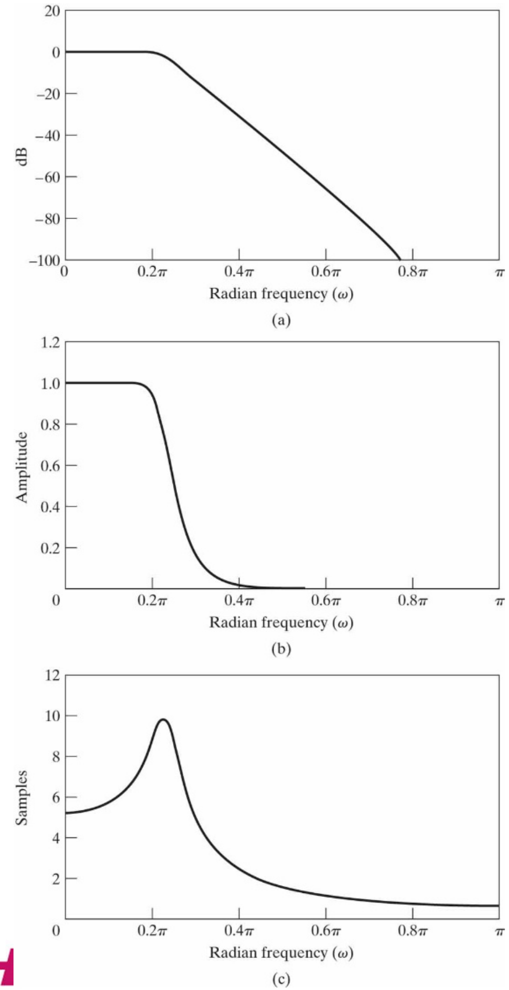
$$|H_c(j2\tan(0.15\pi))| \leq 017783$$

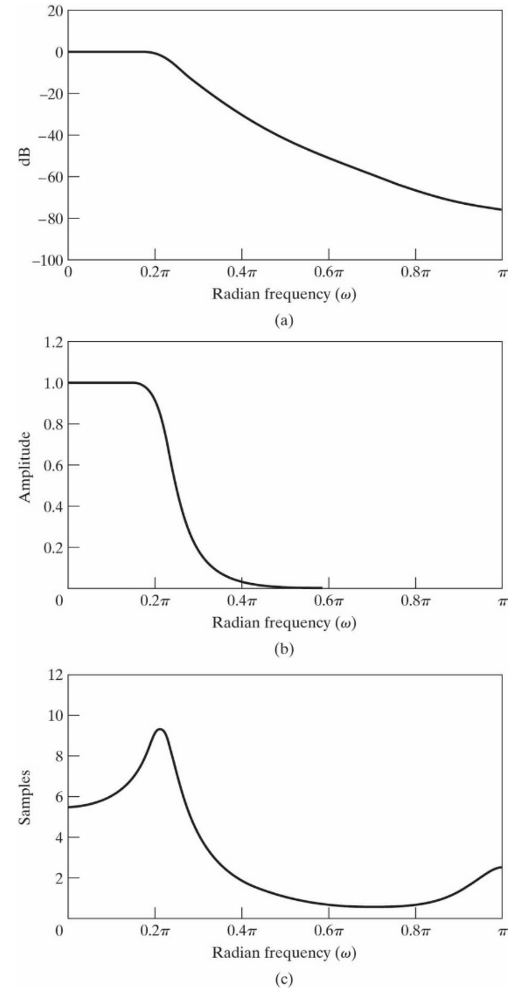
- This is reasonable for bilinear transformation due to lack of aliasing
  - Possible to have the desired stopband edge

- Derive stable system function $H_c(s)$ and apply bilinear transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

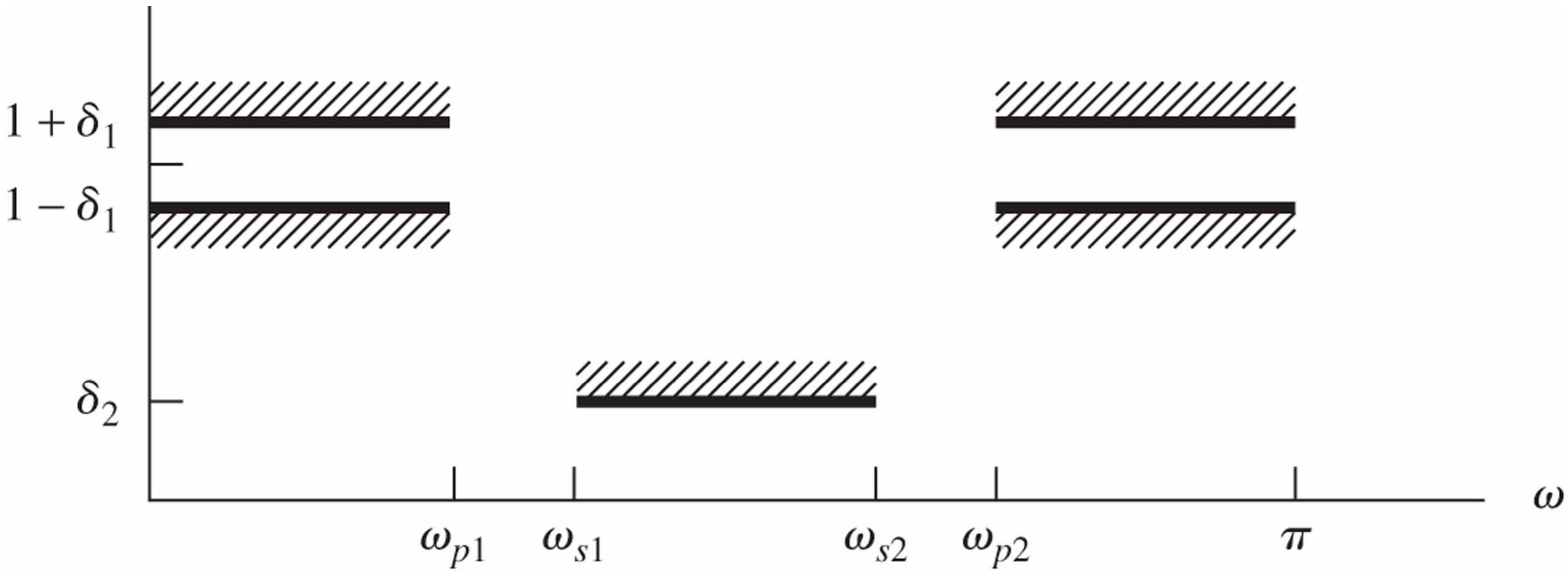
Impulse invariance vs. bilinear transformation





Frequency transformation of lowpass IIR filter

- So far, we have focused on lowpass IIR filter
- How can we implement general bandpass (multiband) filters?



Possible approaches

- Transform from analog multiband filter
  - Acceptable only with bilinear transformation
  - Impulse invariance suffers from aliasing
    ➔ Hard to implement highpass (or multiband) filters

- Transform from discrete-time lowpass filter
  - Works for both impulse invariance and bilinear transformation

Transformation table

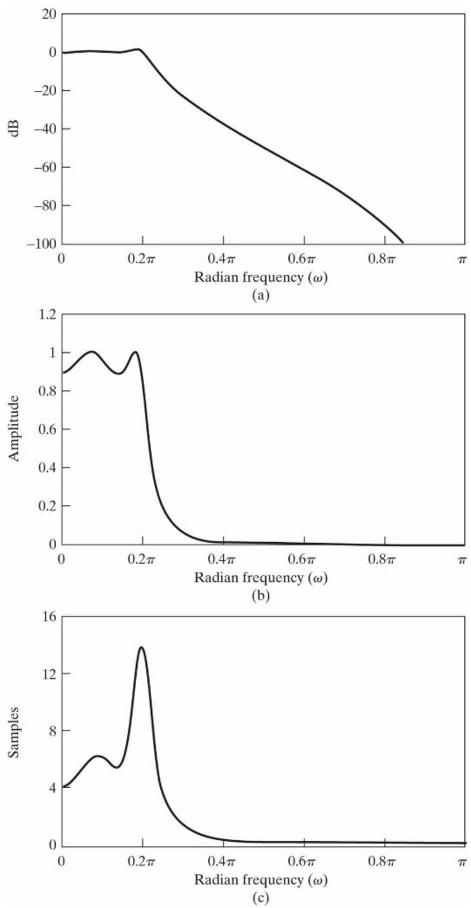
Z-plane of prototype lowpass filter

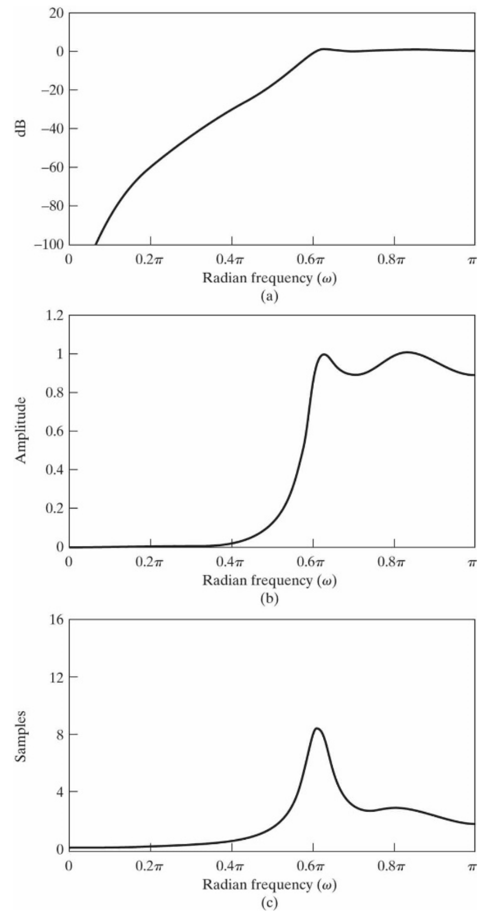
z-plane of desired filter

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency

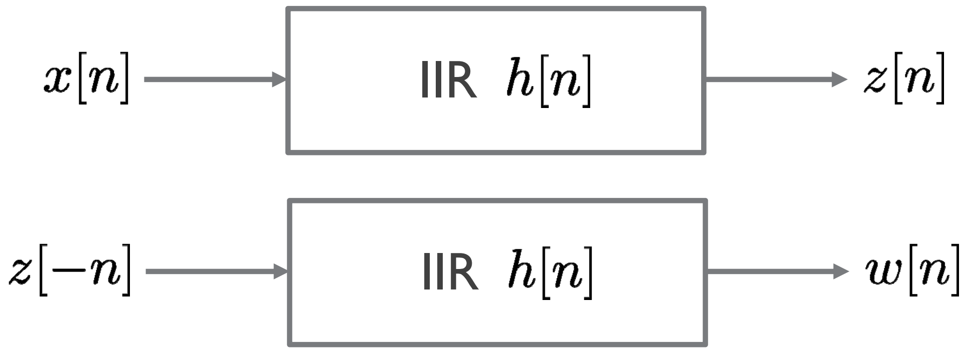
Lowpass to highpass filter transformation





IIR filter with linear phase

- IIR filters generally have nonlinear phases
- Possible to have linear phase IIR filters for non real-time applications



$$y[n] = w[-n]$$

Frequency-domain analysis

- $Z(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

$$z[-n] \overset{\mathcal{DTFT}}{\longleftrightarrow} Z^*(e^{j\omega})$$

$$\begin{aligned} W(e^{j\omega}) &= H(e^{j\omega})Z^*(e^{j\omega}) \\ &= H(e^{j\omega})H^*(e^{j\omega})X^*(e^{j\omega}) \\ &= |H(e^{j\omega})|^2 X^*(e^{j\omega}) \end{aligned}$$

- Since $y[n] = w[-n]$

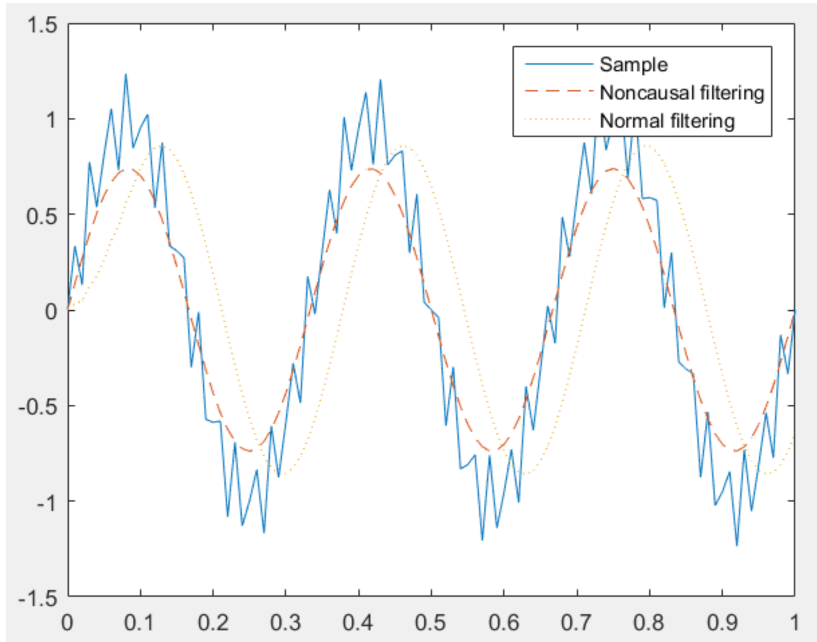
$$Y(e^{j\omega}) = W^*(e^{j\omega}) = |H(e^{j\omega})|^2 X(e^{j\omega})$$

Real number → no phase distortion at all!

Matlab example

```matlab
% Linear phase IIR filter example from Mathworks.com
fs = 100;
t = 0:1/fs:1;
x = sin(2*pi*t*3)+.25*sin(2*pi*t*40);

b = ones(1,10)/10;              % 10 point averaging filter
y = filtfilt(b,1,x);            % Noncausal filtering
yy = filter(b,1,x);             % Normal filtering
plot(t,x,t,y,'--',t,yy,':')
```



FIR Filter Design

FIR filter design

- Design problem: FIR system function

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$h[n] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- Need to find
  - Degree M
  - Filter coefficients b_n (or $h[n]$) for $0 \le n \le M$

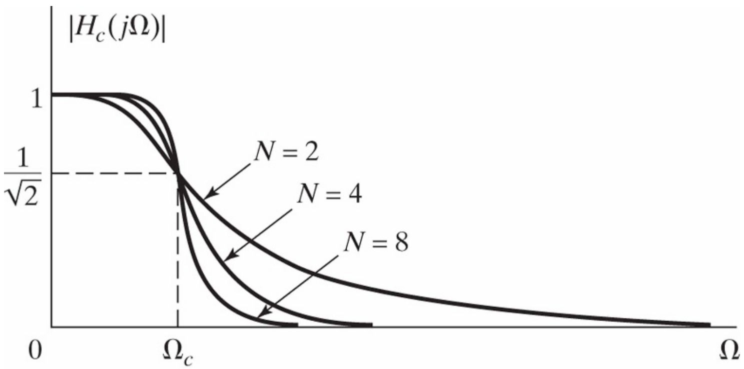
  to approximate the desired frequency response

Lowpass filter as an example

- Ideal lowpass filter $H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < \omega \leq \pi \end{cases}$

$$h[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Discrete-time IIR filter: transform from continuous-time IIR lowpass filter



$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

Get discrete-time filter using either impulse invariance or bilinear transformation

- How to get FIR filter? $h[n]$ is non-causal and infinite!

Design of FIR filter by windowing

- Most straightforward approach
  - Truncate the ideal impulse response by windowing (to have finite length) and do time shifting (to make it causal)

$$h[n] = h_d[n]w[n]$$

Finite length impulse response ($h[n]$)

Infinite length Ideal impulse response ($h_d[n]$)

Finite length window ($w[n]$)

- Use rectangular window for simple truncation

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Lowpass filter example

- Ideal lowpass filter with zero delay

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

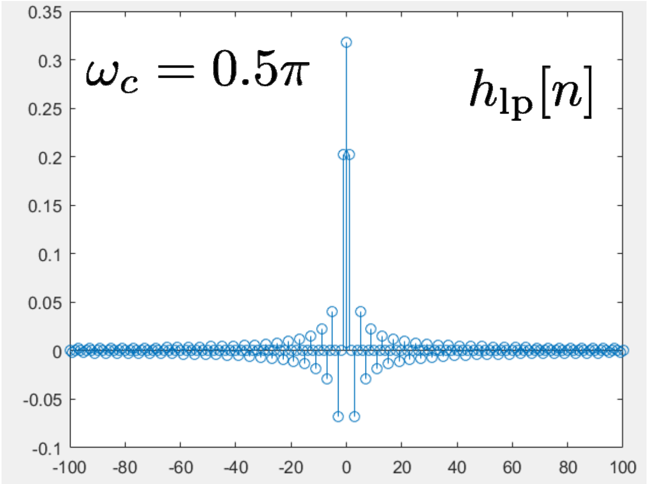
- Ideal lowpass filter with delay

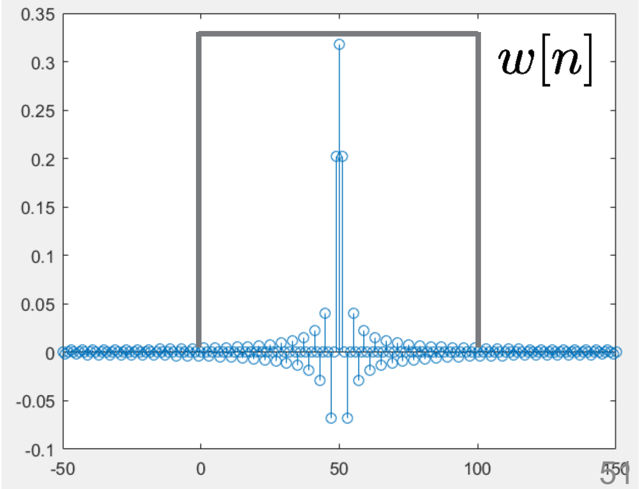
$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -\infty < n < \infty$$

- Ideal lowpass filter with delay and truncation

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -n_1 < n < n_2$$

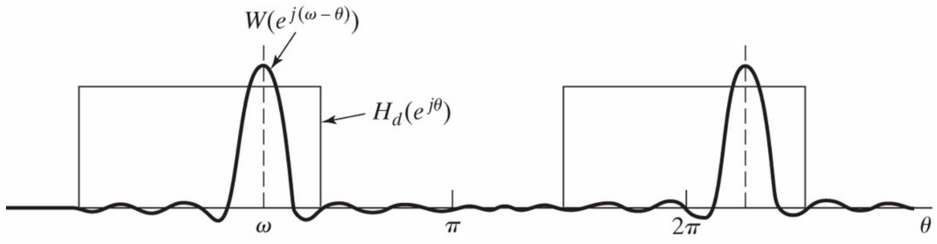




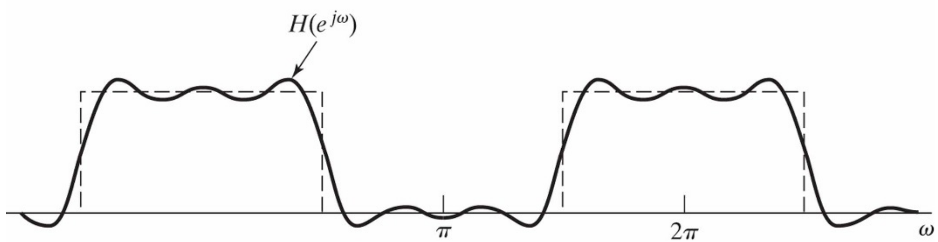
Frequency-domain representation

- From modulation (or windowing) theorem (2.9.7)

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})W(e^{j(\omega-\theta)})d\theta$$



(a)



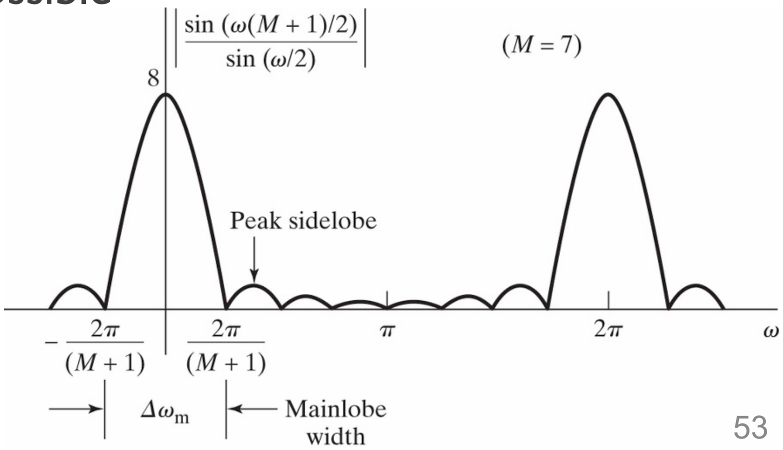
(b)

Tradeoff relationship

- If $w[n] = 1$ for all *n*, no truncation
  ➔ $W(e^{j\omega})$ becomes a periodic impulse train with period 2π
  ➔ $H(e^{j\omega}) = H_d(e^{j\omega})$

- How to choose the window size?
  - As short as possible to minimize implementation computation
  - Have $W(e^{j\omega})$ like an impulse as much as possible
    ➔ Requires long window size

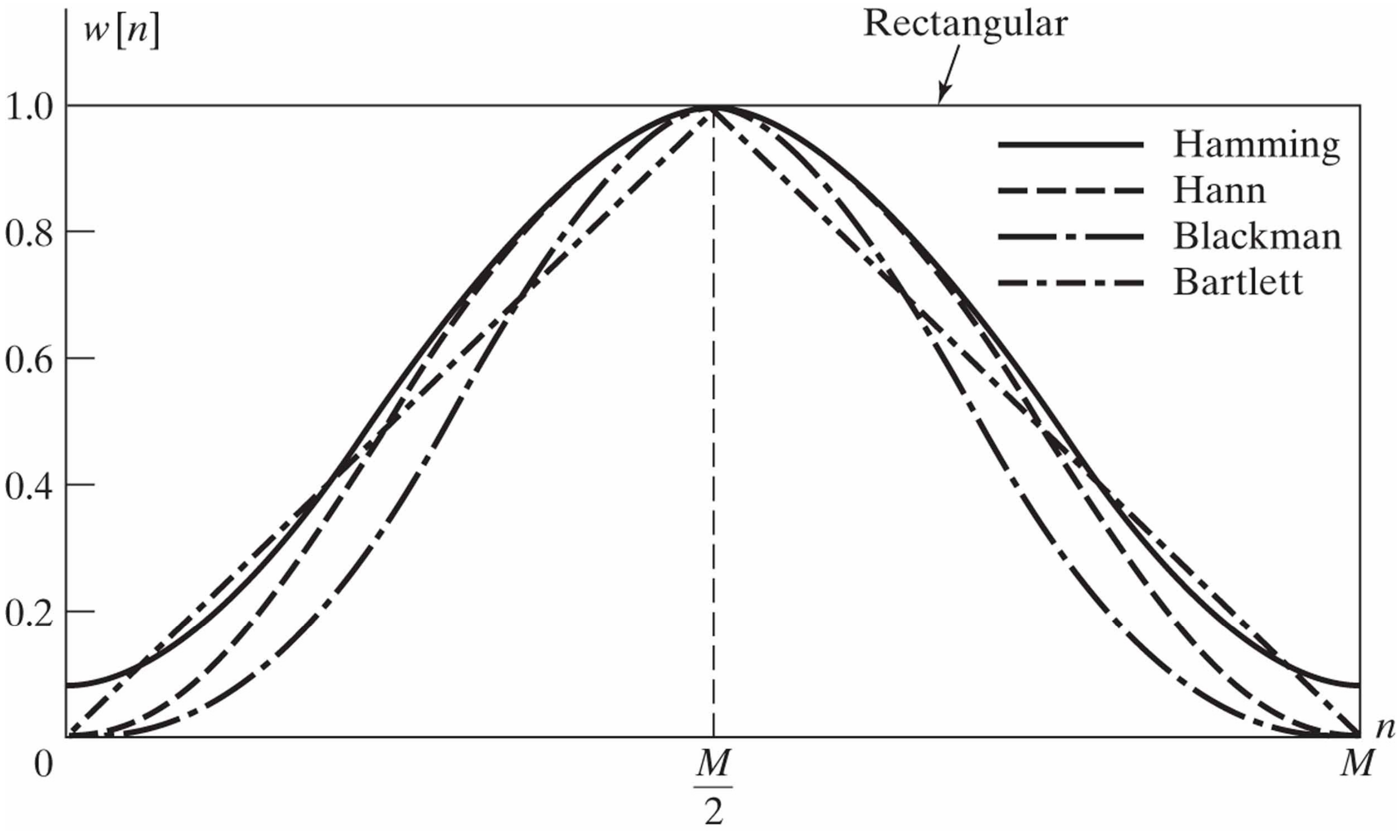
- Abrupt change in window
  ➔ More ripples in frequency-domain



Commonly used windows

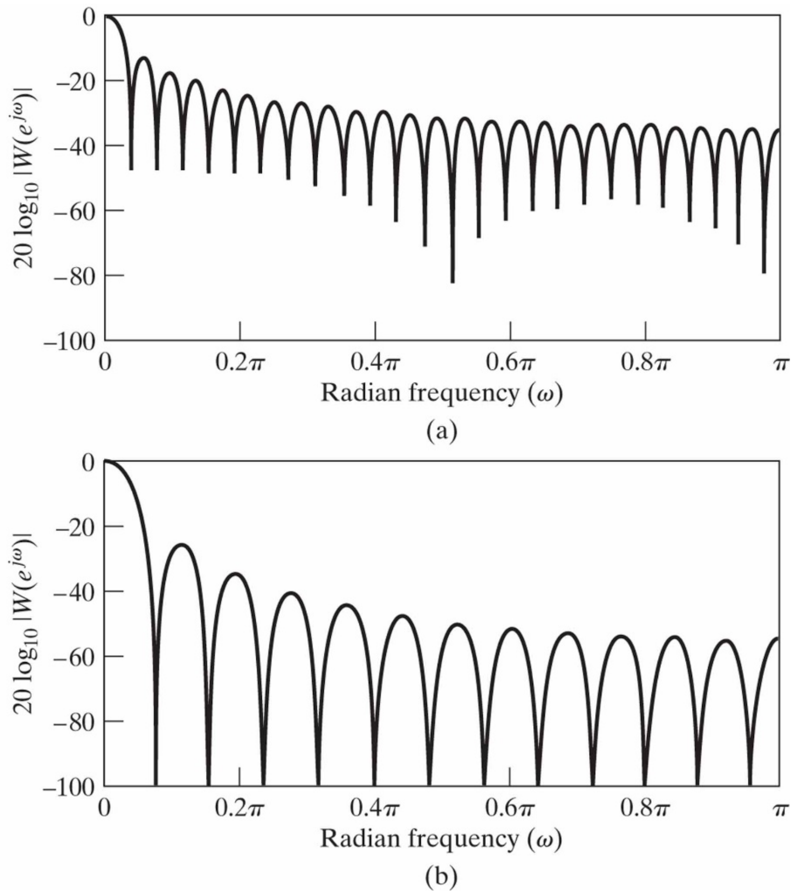
- Check Section 7.5.1
  - Rectangular
  - Bartlett (triangular)
  - Hann
  - Hamming
  - Blackman

Commonly used windows



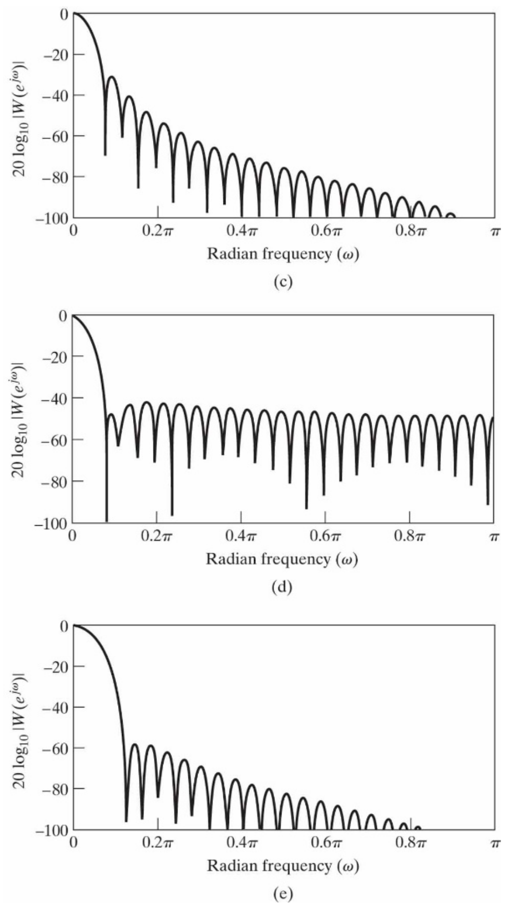
Commonly used windows

Figure 7.30 Fourier transforms (log magnitude) of windows of Figure 7.29 with $M = 50$. (a) Rectangular. (b) Bartlett.



Commonly used windows

Figure 7.30 (continued) (c) Hann. (d) Hamming. (e) Blackman.

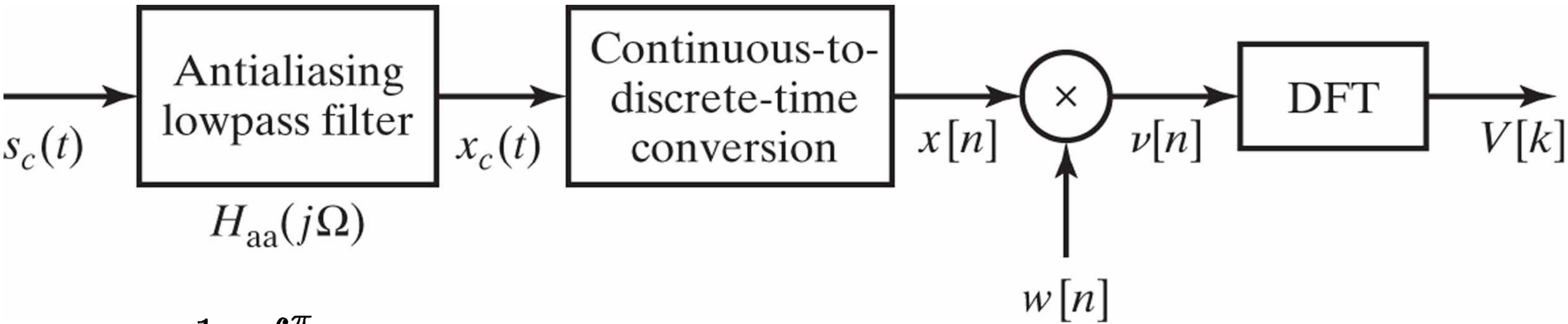


Comparison of standard windows

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Processing steps of Fourier analysis



$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

$$V[k] = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)kn}, \quad k = 0, 1, \ldots, N-1$$

$$= V(e^{j\omega})\,|_{\omega=2\pi k/N}$$

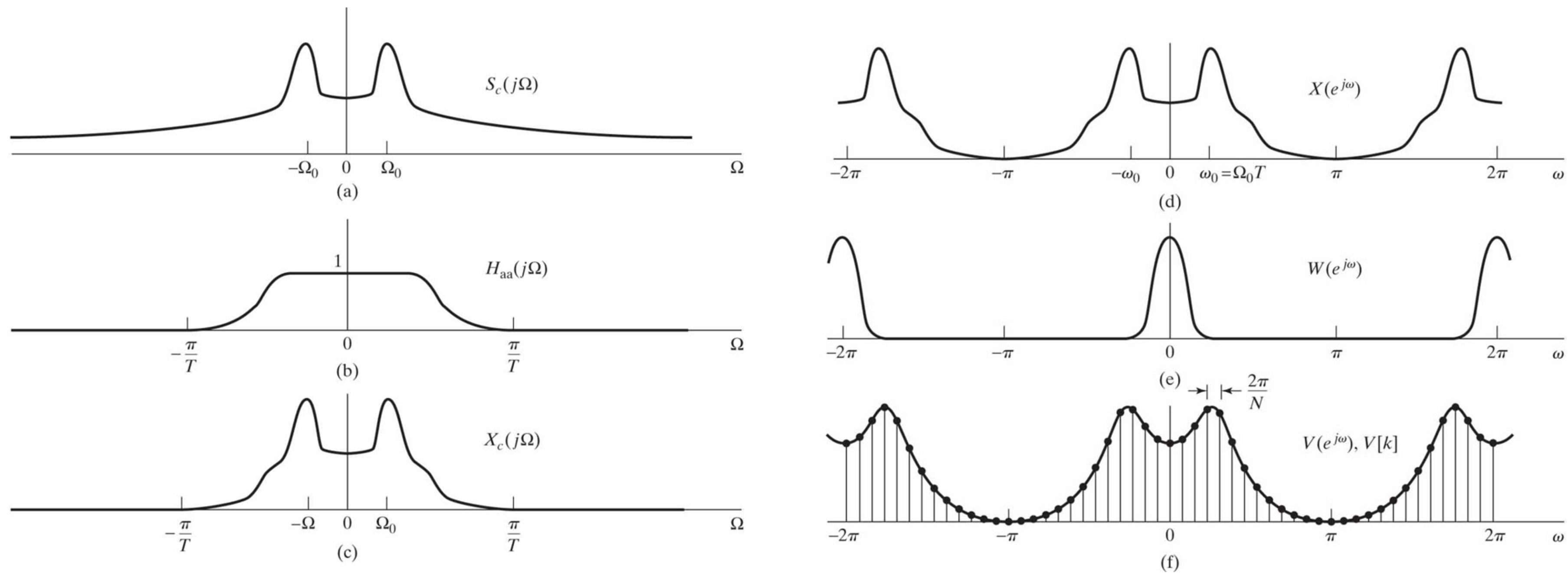
The k-th DFT frequency

$$\omega_k = 2\pi k/N$$

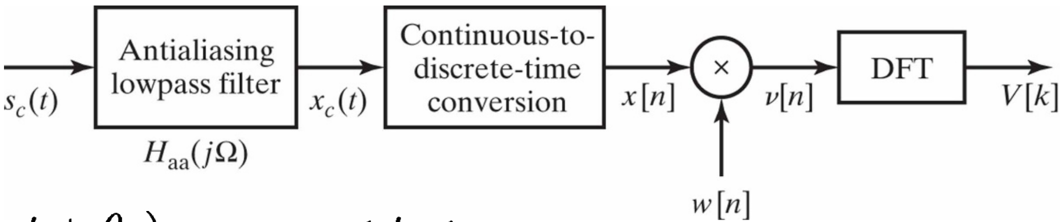
Corresponding continuous-time frequency

$$\Omega_k = \frac{2\pi k}{NT}$$

Illustration



DFT analysis of sinusoidal signals



$$s_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1), \quad -\infty < t < \infty$$

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1), \quad -\infty < n < \infty$$

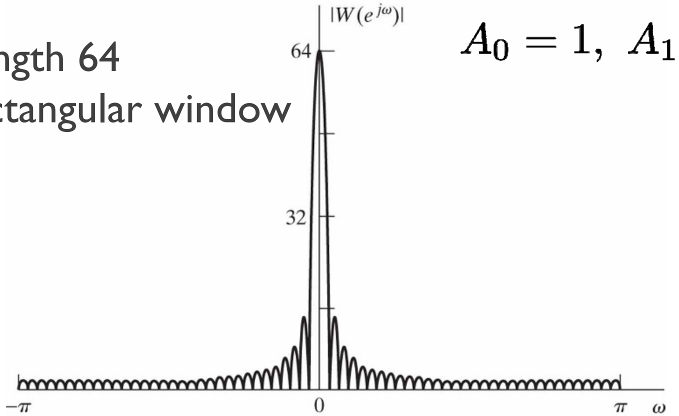
$$\begin{aligned} v[n] &= A_0 w[n] \cos(\omega_0 n + \theta_0) + A_1 w[n] \cos(\omega_1 n + \theta_1) \\ &= \frac{A_0}{2} w[n] e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} w[n] e^{-j\theta_0} e^{-j\omega_0 n} + \frac{A_1}{2} w[n] e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} w[n] e^{-j\theta_1} e^{-j\omega_1 n} \end{aligned}$$

$$V(e^{j\omega}) = \frac{A_0}{2} e^{j\theta_0} W(e^{j(\omega-\omega_0)}) + \frac{A_0}{2} e^{-j\theta_0} W(e^{j(\omega+\omega_0)}) + \frac{A_1}{2} e^{j\theta_1} W(e^{j(\omega-\omega_1)}) + \frac{A_1}{2} e^{-j\theta_1} W(e^{j(\omega+\omega_1)})$$

Effect of windowing

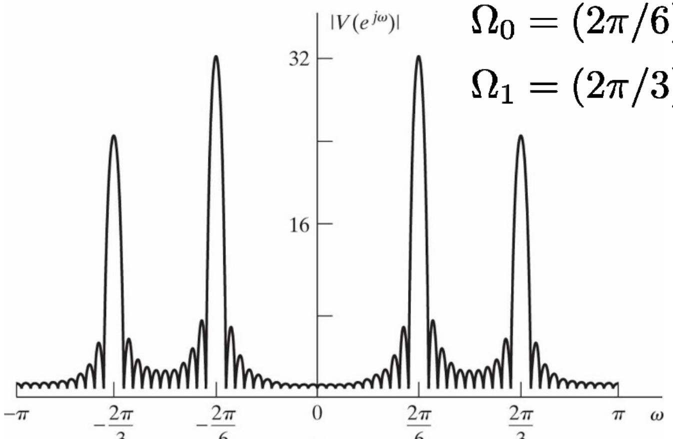
Length 64 rectangular window

$$A_0 = 1,\ A_1 = 0.75,\ \theta_0 = \theta_1 = 0,\ 1/T = 10\ \text{kHz}$$



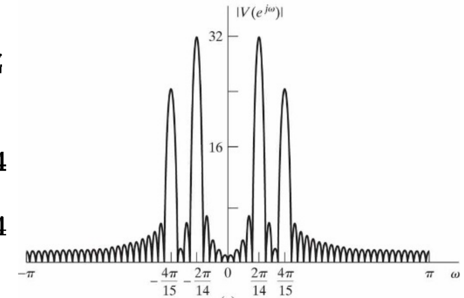
(a)

$$\Omega_0 = (2\pi/6) \times 10^4$$
$$\Omega_1 = (2\pi/3) \times 10^4$$



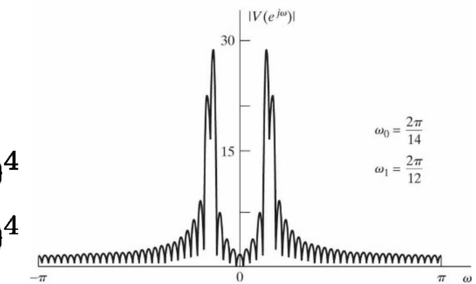
(b)

$$\Omega_0 = (2\pi/14) \times 10^4$$
$$\Omega_1 = (4\pi/15) \times 10^4$$



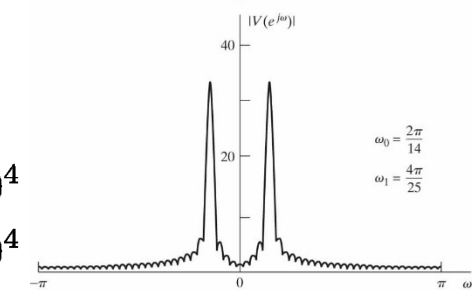
(c)

$$\Omega_0 = (2\pi/14) \times 10^4$$
$$\Omega_1 = (2\pi/12) \times 10^4$$



(d)

$$\Omega_0 = (2\pi/14) \times 10^4$$
$$\Omega_1 = (4\pi/25) \times 10^4$$



(e)

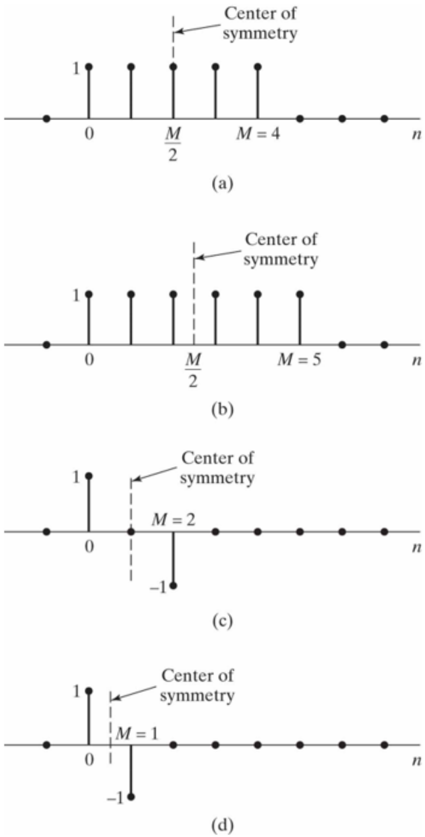
Effect of windowing

- Two primary effects
  - Reduced resolution for close frequencies
    ➔ Related to width of main lobe
  - Leakage from one frequency component to another
    ➔ Related to relative amplitude ratio of main lobe to side lobes

- Need to carefully select window depending on applications

Generalized linear phase FIR filter by windowing

- Recall four classes of FIR systems with generalized linear-phase
  - Type I
    - Symmetric: $h[n] = h[M-n]$
    - M even
  - Type II
    - Symmetric: $h[n] = h[M-n]$
    - M odd
  - Type III
    - Antisymmetric: $h[n] = -h[M-n]$
    - M even
  - Type IV
    - Antisymmetric: $h[n] = -h[M-n]$
    - M odd



Generalized linear phase FIR filter by windowing

- Often aim at designing a causal system with a generalized linear phase
  - Stability is not a problem with FIR systems
- With (anti-)symmetric (possibly infinite length) impulse response,

$$h_d[M-n] = h_d[n]$$

Symmetric at *M*/2

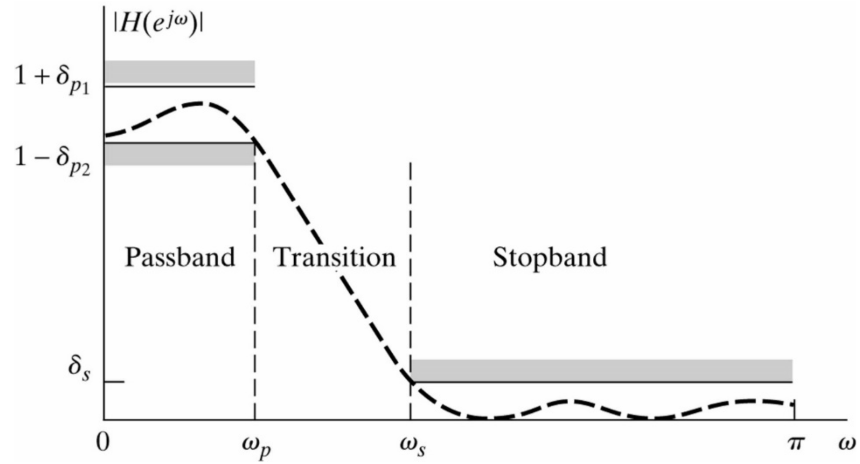
choose windows being symmetric at *M*/2

$$w[n] = \begin{cases} w[M-n], & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

- Truncated filter $h[n] = h_d[n]w[n]$ still (anti-)symmetric
  ➔ Generalized linear phase!!!

FIR filter design procedure

- Specifications to meet
  - Transition bandwidth: $\triangle\omega = \omega_s - \omega_p$
  - Ripple levels: $\delta_s, \ \delta_p$
- Window design method
  - Choose window shape
  - Adjust window length *M*
    ➔ One parameter to adjust
- Need to perform trial and error
  ➔ Not a good method



Kaiser window

- Formalization of window design method
  - No need for trial and error
  - Easy way to find the trade-off between the main-lobe width and side-lobe area
  - Design with two parameters: M and β
  - Can control the tradeoff between side-lobe amplitude and main-lobe width
    - This was not possible for previous windows

- Kaiser window expression

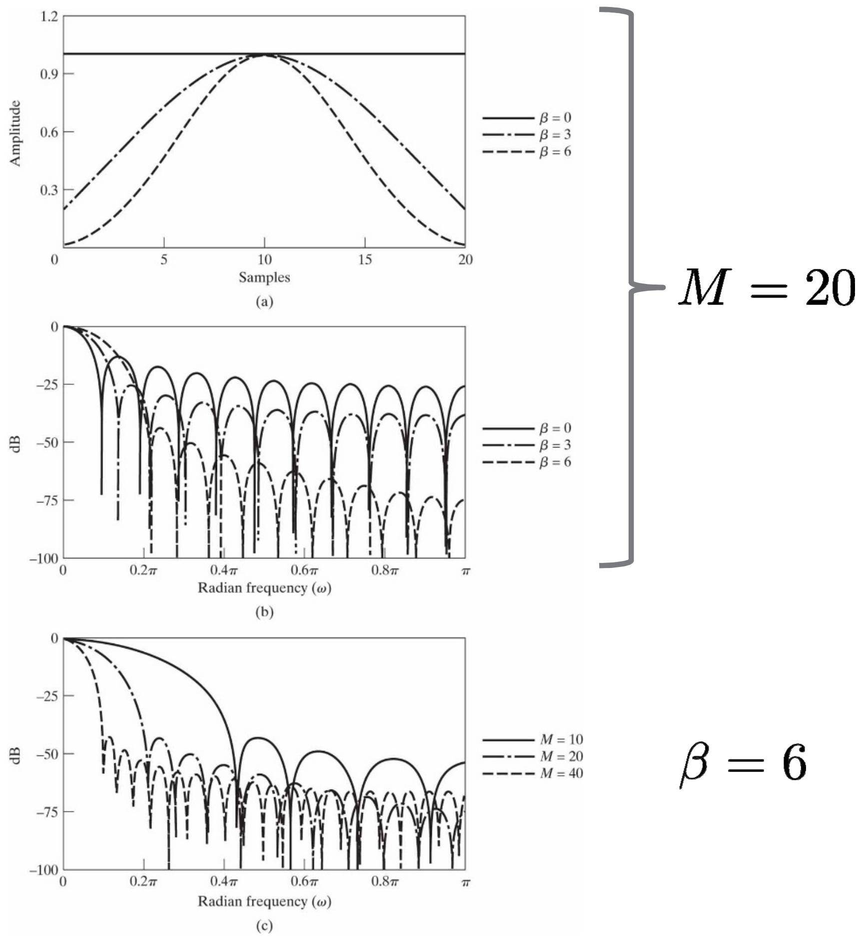
Zeroth-order Bessel function of the first kind

$$w[n] = \begin{cases} I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]/I_0(\beta), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

with $\alpha = M/2$ and $\beta \geq 0$

Kaiser window

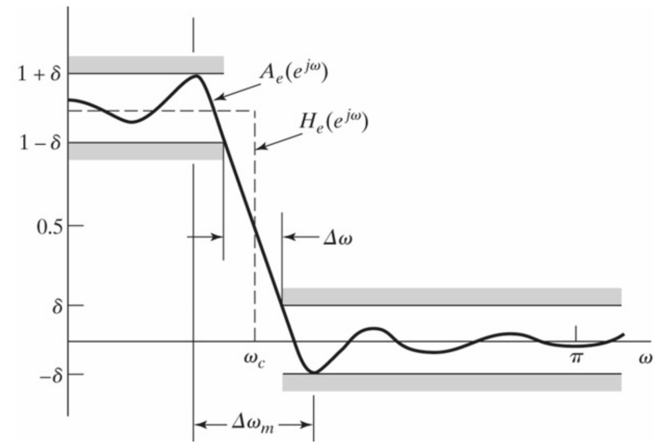
- Becomes rectangular window when $\beta = 0$



Design FIR filter by Kaiser window

- Calculate M and β to meet the filter specifications
  - The peak approximation error δ is determined by β
- Define $A = -20\log_{10}\delta$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0, & A < 21 \end{cases}$$



- Passband/stopband frequencies also defined by δ

$$|H(e^{j\omega_p})| \ge 1-\delta, \quad |H(e^{j\omega_s})| \le \delta$$

  - Transition width becomes $\Delta\omega = \omega_s - \omega_p$

- Possible to show $M = \dfrac{A-8}{2.285\Delta\omega}$ to satisfy the specifications

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Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Matlab examples

```
% Butterworth filter example
fc = 300;    % cutoff frequency (in Hz)
fs = 1000;  % sampling frequency (in Hz)
[b,a] = butter(6,fc/(fs/2));
freqz(b,a)

dataIn = randn(1000,1);
dataOut = filter(b,a,dataIn);

figure(2)
plot(1:1000,dataIn,1:1000,dataOut,'r')
legend('Random samples','Filtered samples')
```

Matlab examples

```
% designfilt example

lpFilt = designfilt('lowpassfir','PassbandFrequency',0.25, ...
'StopbandFrequency',0.35,'PassbandRipple',0.5, ...
'StopbandAttenuation',65,'DesignMethod','kaiserwin');
fvtool(lpFilt)
dataIn = rand(1000,1);
dataOut = filter(lpFilt,dataIn);
plot(1:1000,dataIn,1:1000,dataOut,'r')
```