Digital Signal Processing

POSTECH
Department of Electrical Engineering
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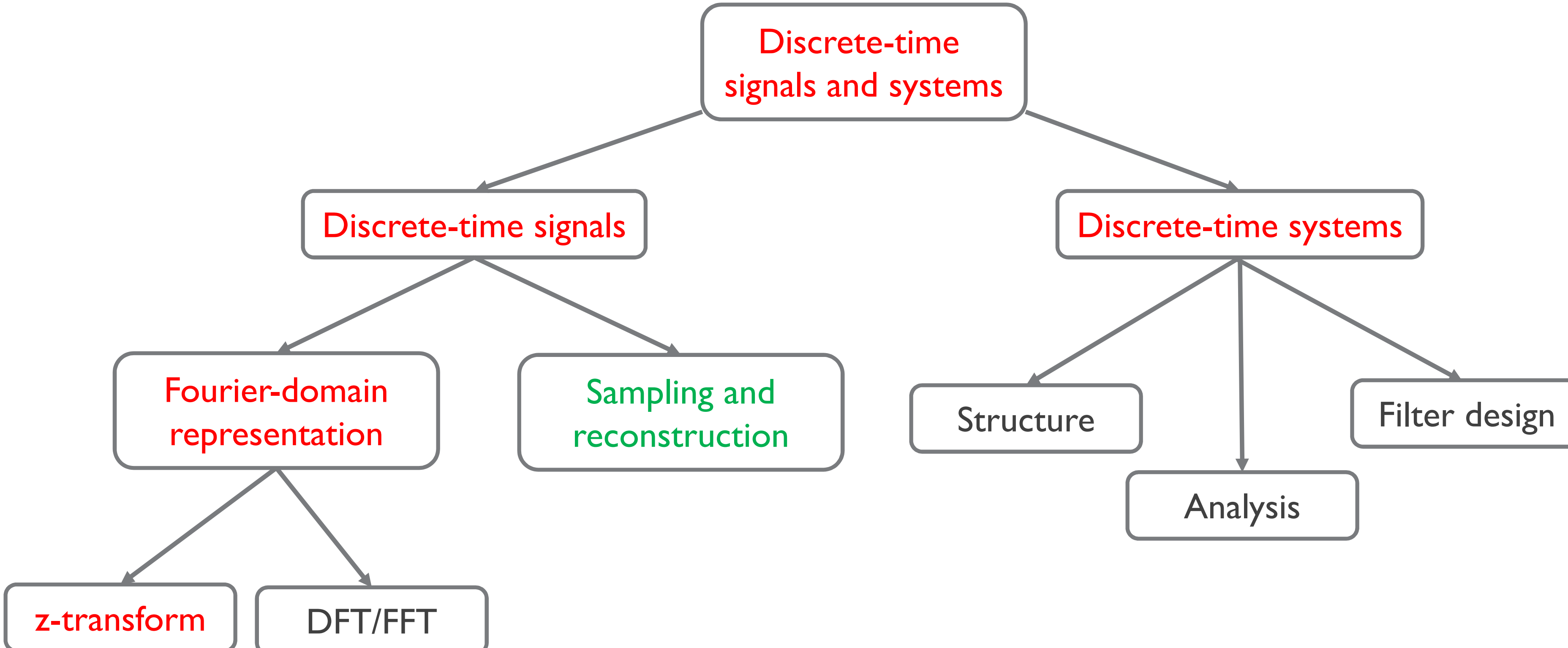
Course at glance



Two-stages representation

- Mathematically
  - Impulse train modulator
  - Conversion of the impulse train into a sequence

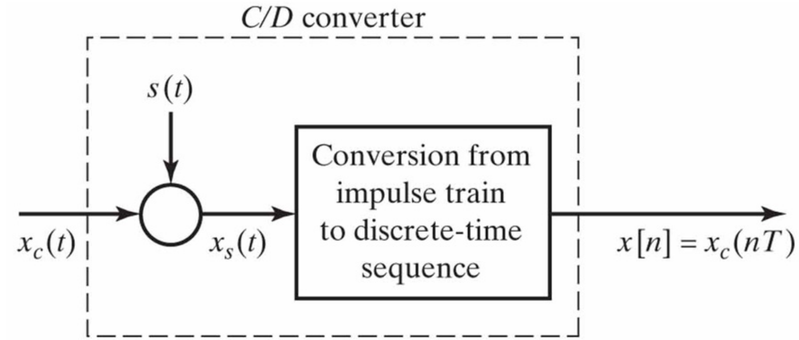
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

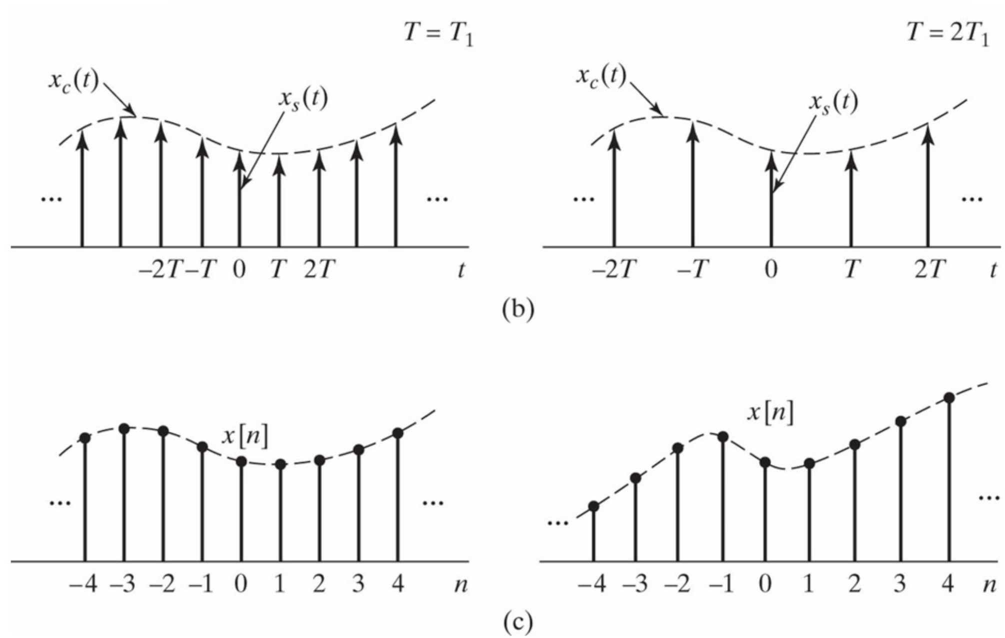
$$x_s(t) = x_c(t)s(t)$$

$$= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x[n] = x_c(nT)$$





Frequency-domain representation of sampling

- Fourier transform of impulse train is also the periodic impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathcal{F}}{\longleftrightarrow} S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$\Omega_s = \frac{2\pi}{T}$$

- Fourier transform of impulse train-modulated signal

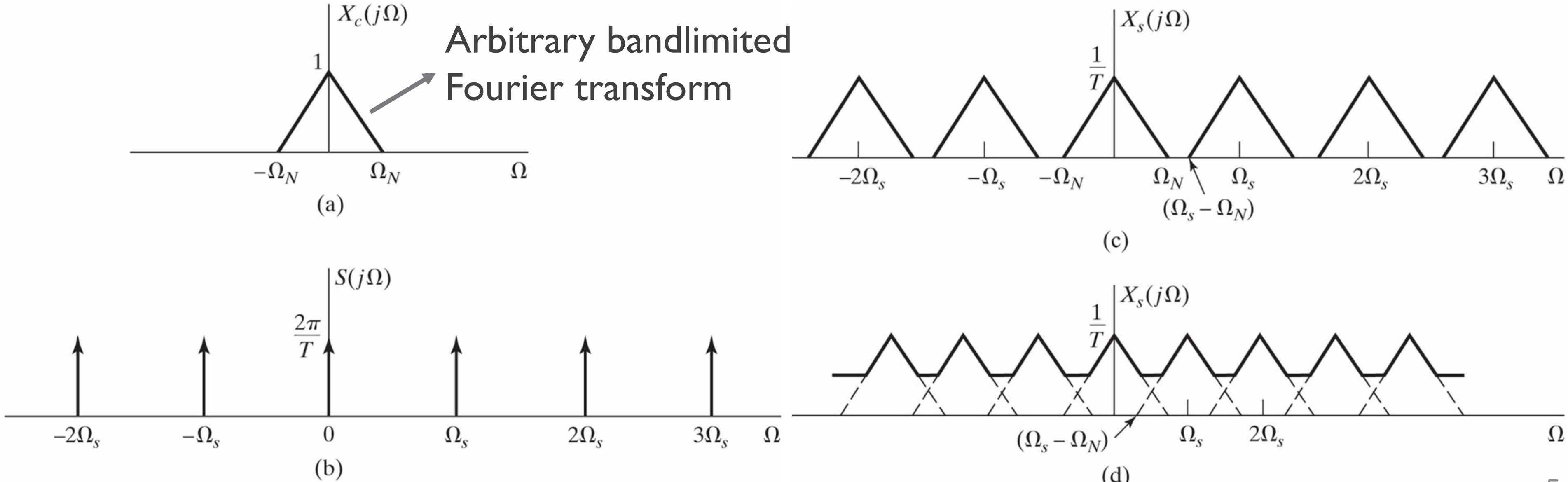
$$x_s(t) = x_c(t)s(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

Continuous-variable convolution

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Close look into Fourier transform of sampled signal

- Recall $X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$
  - Consist of periodic repeated copies of $X_c(j\Omega)$
  - Copies are shifted by integer multiples of sampling frequency Ω_s



Nyquist-Shannon sampling theorem

- Given a bandlimited signal $x_c(t)$ with

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \;\; n = 0, \pm 1, \pm 2, \ldots$$

if $\Omega_s = \dfrac{2\pi}{T} \geq 2\Omega_N$

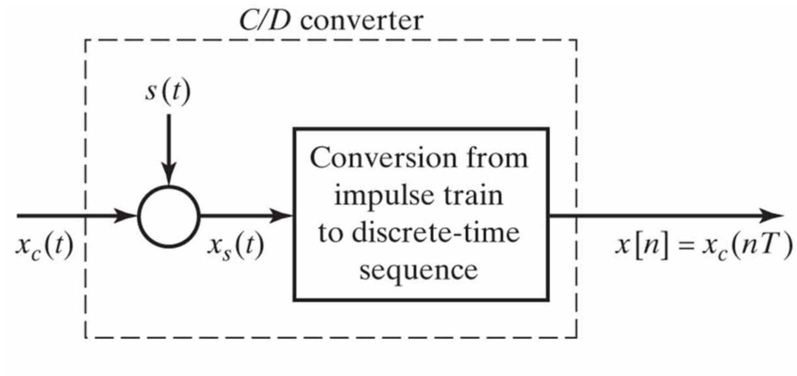
- Ω_N is called Nyquist frequency
- $2\Omega_N$ is called Nyquist (sampling) rate

Fourier transform of x[n]

- From $x_c(t)$ to $x[n]$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$



(a)

- From $X_c(j\Omega)$ to $X(e^{j\omega})$
  - By taking continuous-time Fourier transform

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}$$

  - By taking discrete-time Fourier transform $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Fourier transform of x[n] (continue)

- Relation between $X_c(j\Omega)$ and $X(e^{j\omega})$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

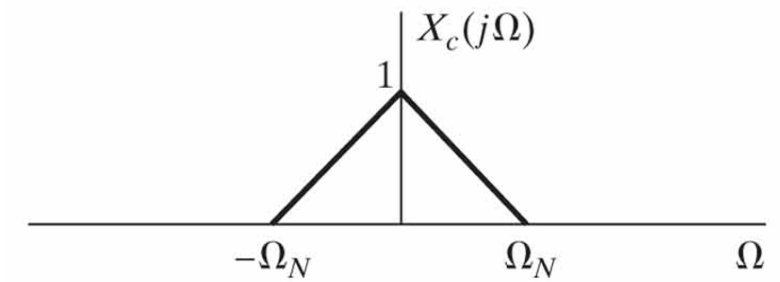
$$\Rightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with $\omega = \Omega T$
  - It can also be thought as frequency axis normalization
  - Sampling frequency $\Omega_s = 2\pi/T$ ➔ $\omega_s = 2\pi$
  - Sampling frequency always mapped to $\omega_s = 2\pi$ in DTFT

Reconstruction

Requirement for reconstruction

- Based on Nyquist sampling theorem, a signal can be exactly recovered from its samples when
  - The signal is bandlimited $X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$



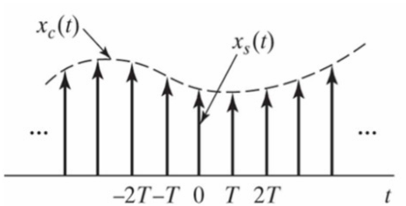
  - Sampling frequency is large enough $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$
  - + knowledge of sampling period to recover the signal
    ➔ To determine bandwidth of lowpass filter

Reconstruction steps

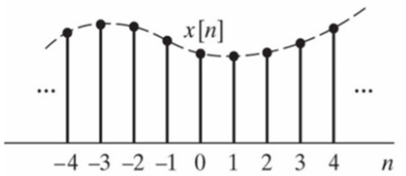
- (1) Given $x[n]$ and T, form a continuous-time impulse train $x_s(t)$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

➔ the n-th sample is associated with the impulse at $t=nT$



- (2) $x_s(t)$ is filtered by an ideal lowpass continuous-time filter $h_r(t)$ with frequency response $H_r(j\Omega)$



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT)$$

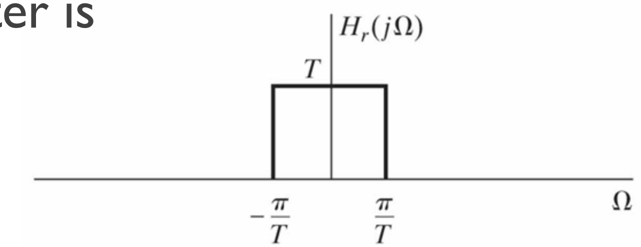
Mathematical expression of reconstruction

- Assume the cutoff frequency of ideal lowpass filter is

$$\Omega_c = \Omega_s/2 = \pi/T$$

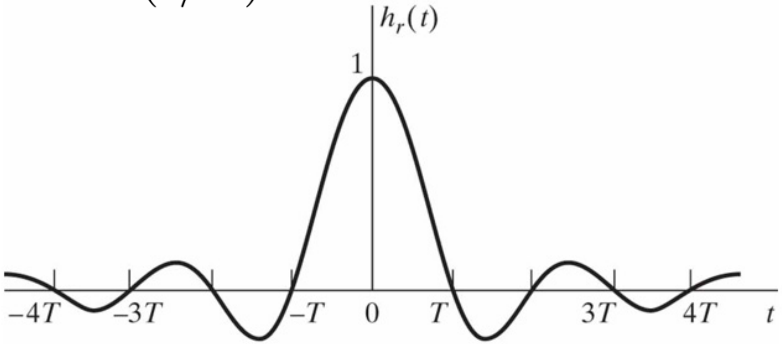
  - Any cutoff frequency works as long as

$$\Omega_N \leq \Omega_c \leq \Omega_s - \Omega_N$$



- Impulse response of the ideal lowpass filter is

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \mathrm{sinc}(t/T)$$



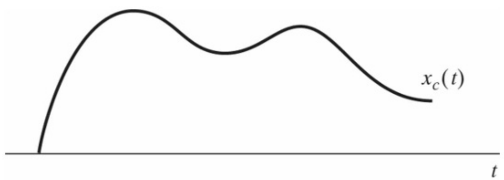
$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

Mathematical expression of reconstruction

- Reconstructed signal becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

➔ Is this the same as $x_c(t)$?



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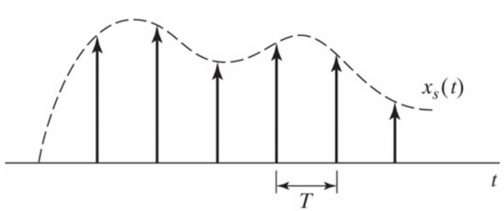

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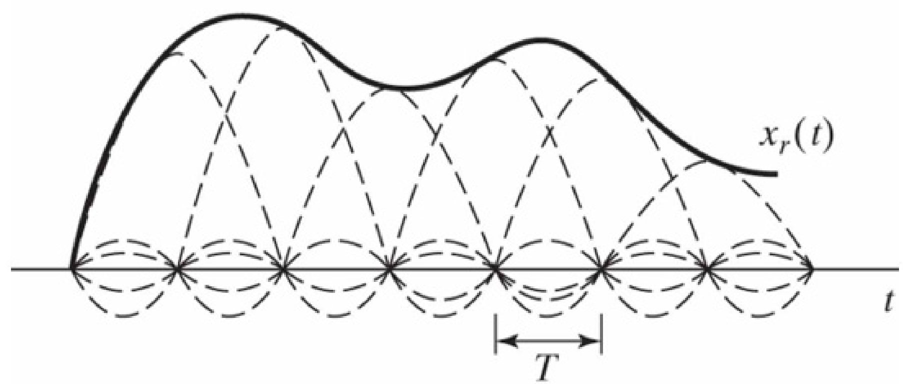
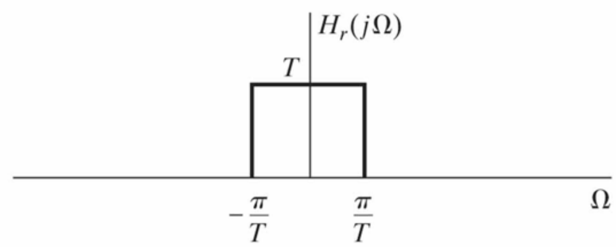


(c)

Ideal D/C converter in frequency domain

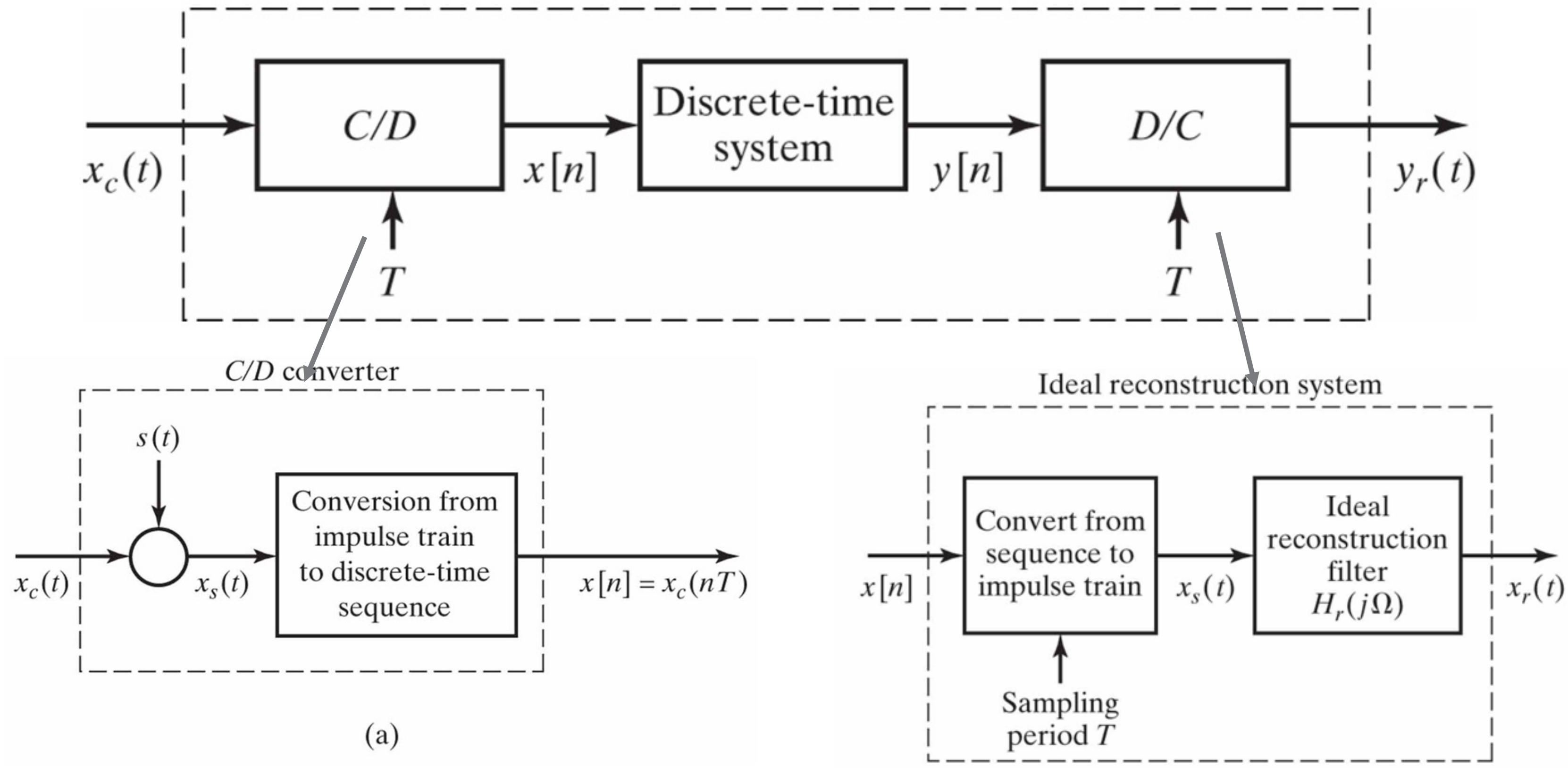
- Recall $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t-nT)$



- $X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]H_r(j\Omega)e^{-j\Omega Tn} = H_r(j\Omega)X(e^{j\Omega T})$
- $X(e^{j\Omega T})$: frequency-scaled version of $X(e^{j\omega})$ with $\omega = \Omega T$
- $H_r(j\Omega)$ selects the base period of the periodic $X(e^{j\Omega T})$ and compensate for $1/T$ scaling from sampling

Discrete-Time Processing of Continuous-Time Signals

Overall block diagram



- Overall system is continuous-time processing
- Continuous-time processing of discrete-time signals also possible

Output signal

- Necessary conditions
  - The discrete-time system is LTI
  - Continuous-time signal $x_c(t)$ is bandlimited
  - Sampling rate Ω_s is at or above the Nyquist rate $2\Omega_N$

- If all conditions are satisfied, the output signal becomes

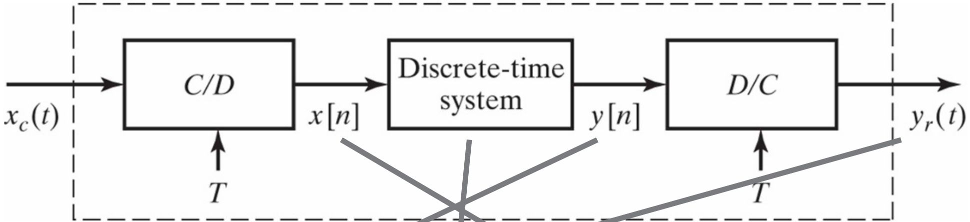
$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega)$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Cutoff frequency of ideal lowpass filter

Detailed steps



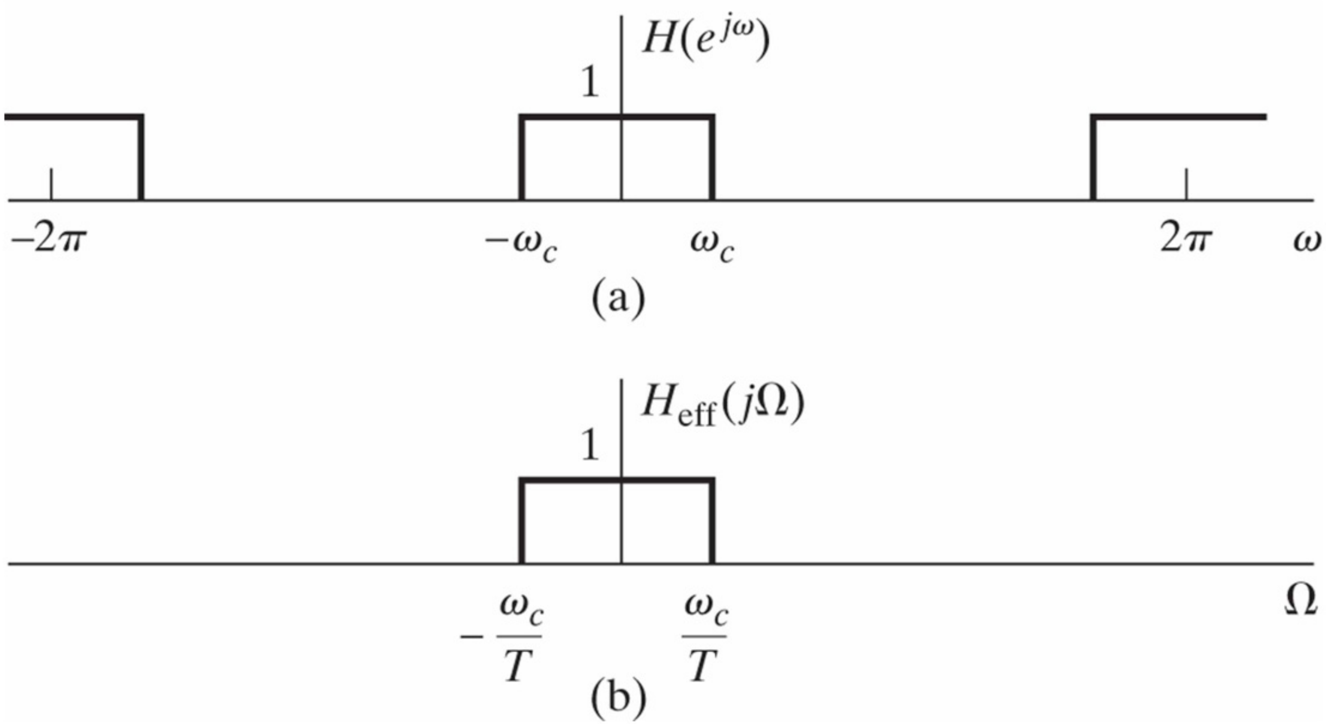
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- If the system is LTI: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- $Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\Omega - \frac{2\pi k}{T}\right)\right]$

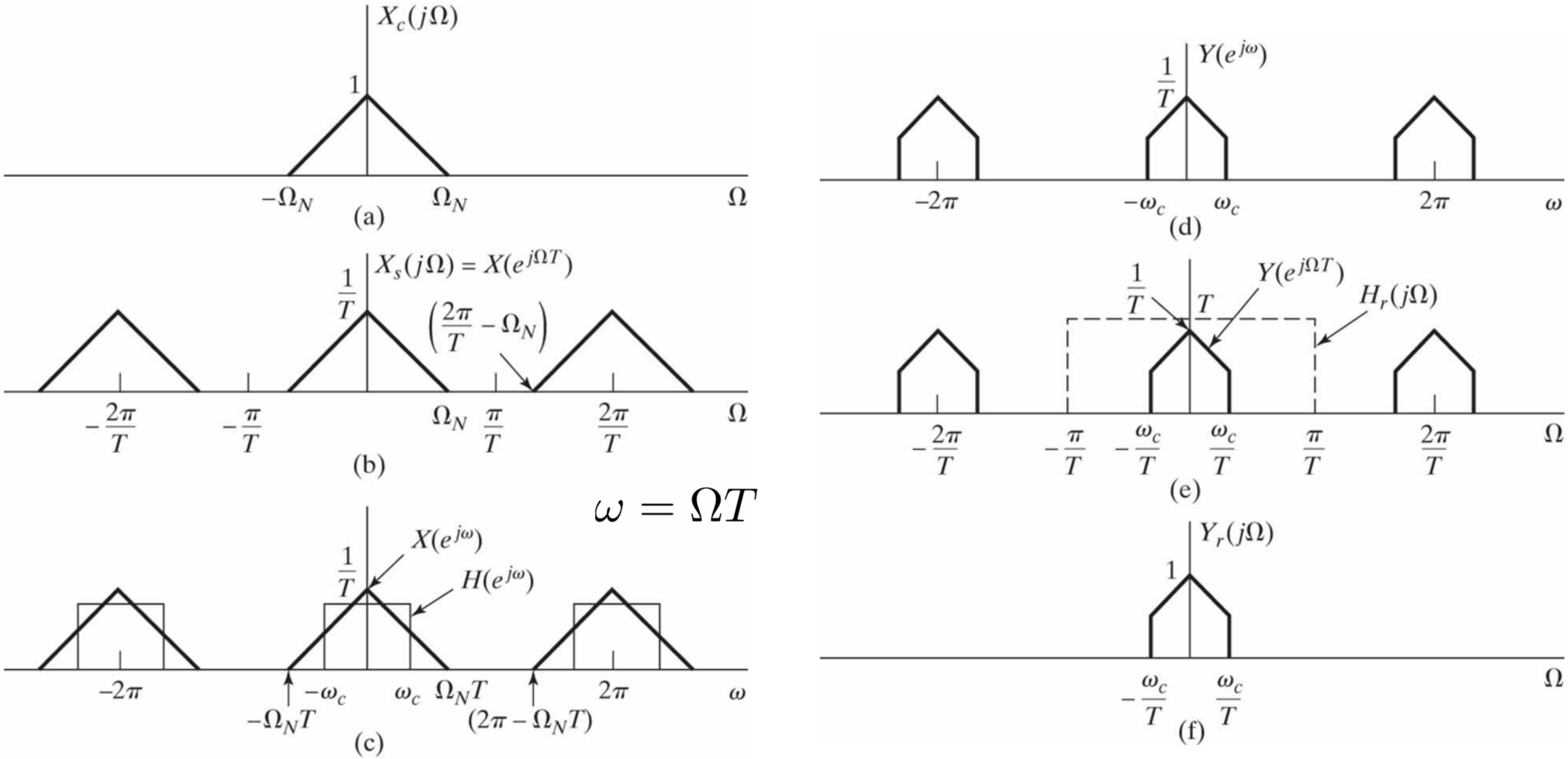
  Ideal lowpass filter in D/C
- If $x_c(t)$ is bandlimited, i.e., $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, and the sampling rate is at or above the Nyquist rate

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Lowpass filtering example

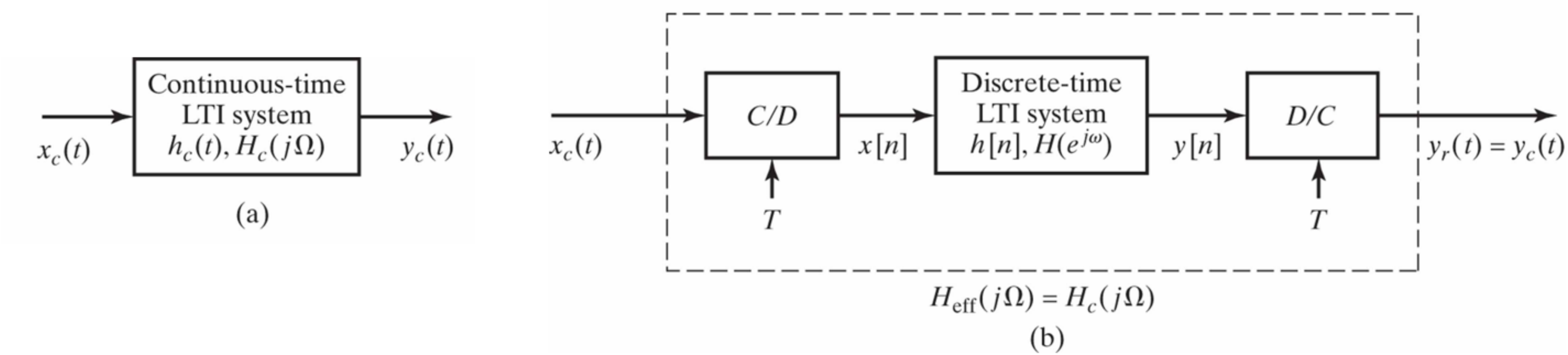


Lowpass filtering example



$$\omega = \Omega T$$

Impulse invariance



- Want to implement the continuous-time impulse response $h_c(t)$ using discrete-time system $h[n]$ or vise versa

- How to design $h[n]$ based on $h_c(t)$?

Impulse invariance

- Recall $H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$

- We want to have $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

$$\Rightarrow \quad H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi$$

- In time-domain: $h[n] = T h_c(nT)$

$$H(e^{j\omega}) = T\frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$= H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Because $H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$

Impulse invariance example

- How to obtain an ideal lowpass discrete-time filter with cutoff frequency $\omega_c < \pi$ from a continuous-time ideal lowpass filter?

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- Define the corresponding discrete-time impulse response as

$$h[n] = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n} \text{ with } \omega_c = \Omega_c T$$

Ideal discrete-time lowpass filter of $H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$

Changing Sampling Rate Using Discrete-Time Processing

Resampling

- Sampling with sampling period *T*: $x[n] = x_c(nT)$

- Often necessary to change the sampling rate of a discrete-time signal

$$x_1[n] = x_c(nT_1), \text{ with } T \neq T_1$$

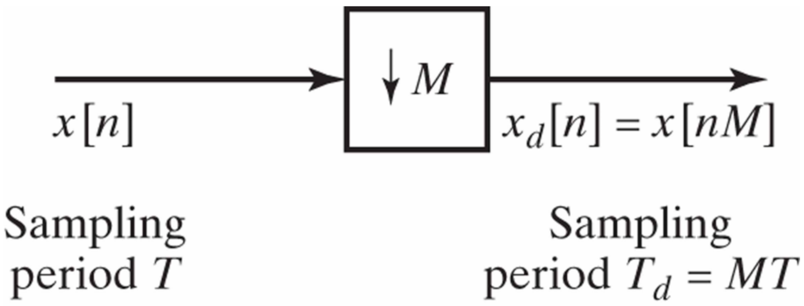
  - Resizing digital images
  - Video/audio conversion

- Direct approach is to reconstruct $x_c(t)$ from $x[n]$ and resample with sampling period T_1
  - Not a practical approach due to non-ideal hardware
  - Near-ideal filters are $$$$$$$

- Can we change the sampling rate by only dealing with discrete-time operations? YES!

Downsampling

Decreasing sampling rate by integer factor



- Usually called "downsampling"
- Sampling rate can be reduced by "sampling" the original sampled sequence
  - Original sampled sequence $x[n] = x_c(nT)$
  - New "sampled" sequence $x_d[n] = x[nM] = x_c(nMT)$
  - Keep one sample out of every *M* samples

  ➔ Operation called "compressor"
- The new sequence $x_d[n]$ is identical to the sequence obtained from $x_c(t)$ with the sampling period $T_d = MT$

Is reconstruction possible?

- Original sampling rate $\Omega_s = 2\pi/T$

- If $X_c(j\Omega) = 0 \text{ for } |\Omega| \geq \Omega_N,\ x_c(t)$ can be reconstructed from $x_d[n]$ if

$$\pi/T_d = \pi/(MT) \geq \Omega_N \Rightarrow 2\pi/T_d \geq 2\Omega_N$$

- Sampling rate can be reduced to $1/M$ without aliasing if the original sampling rate *T* is at least M times the Nyquist rate

Frequency-domain representation

- DTFT of $x[n] = x_c(nT)$ is

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

- DTFT of $x_d[n] = x[nM] = x_c(nT_d)$ with $T_d = MT$

$$X_d(e^{j\omega}) = \frac{1}{T_d}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T_d} - \frac{2\pi r}{T_d}\right)\right]$$

$$= \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right]$$

Frequency-domain representation

- We can write $r = i + kM$ for $-\infty < k < \infty$ and $0 \le i \le M-1$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right]$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right]\right\}$$

- Using DTFT of x[n]

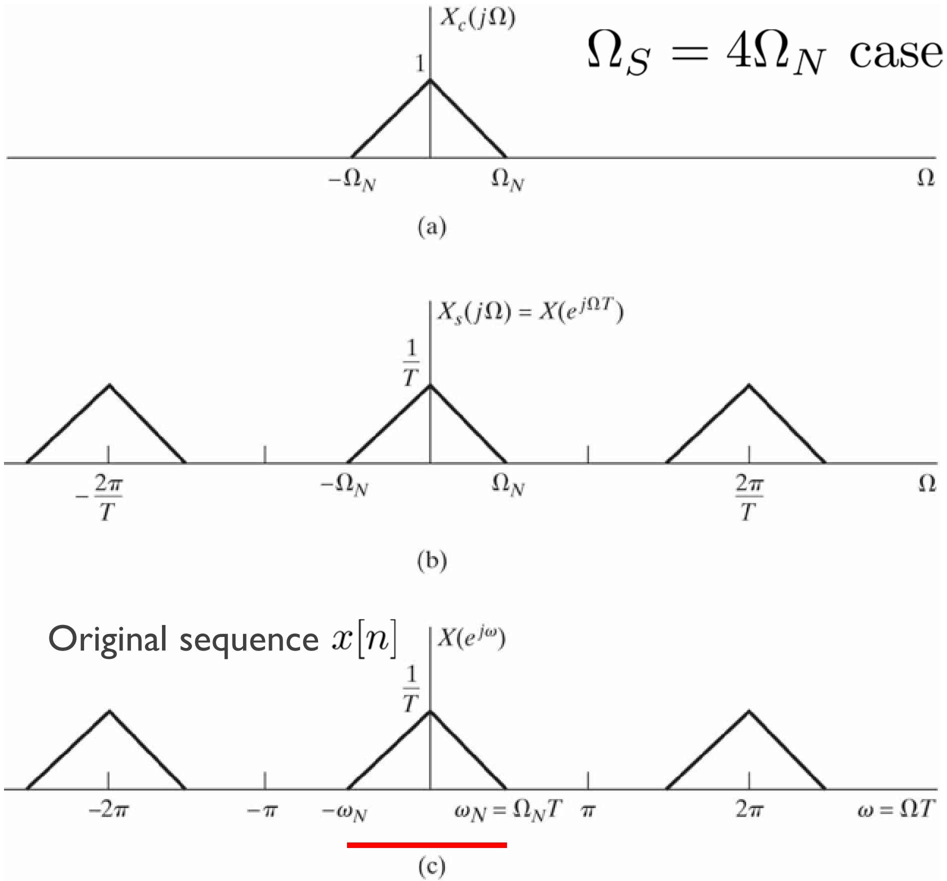
$$X(e^{j(\omega-2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right]$$

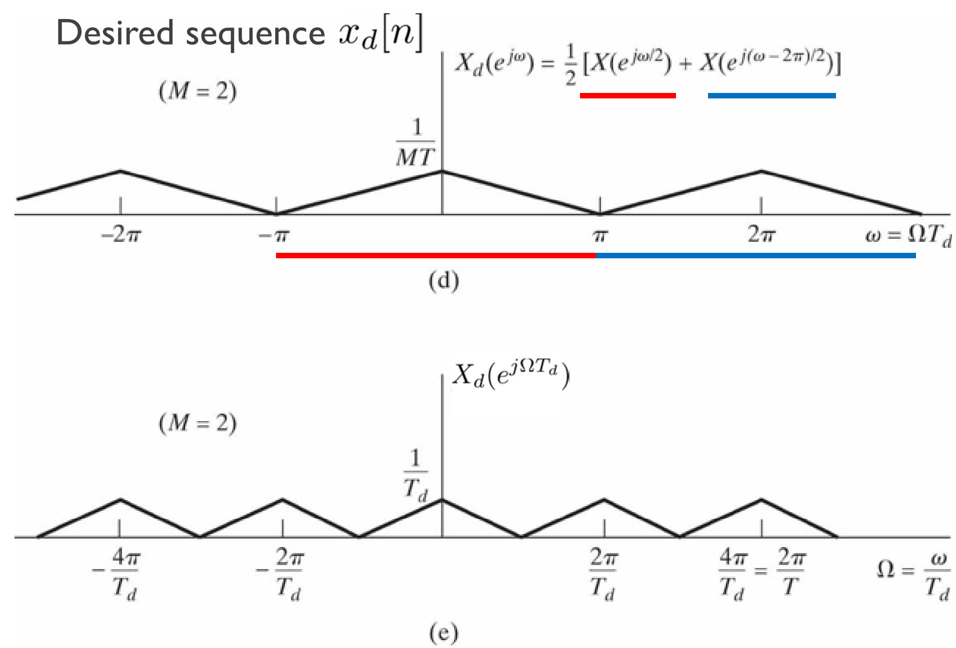
- We have $X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega-2\pi i)/M})$

Scaled-copies of $X(e^{j\omega})$

Example – no aliasing

$\Omega_S = 4\Omega_N$ case

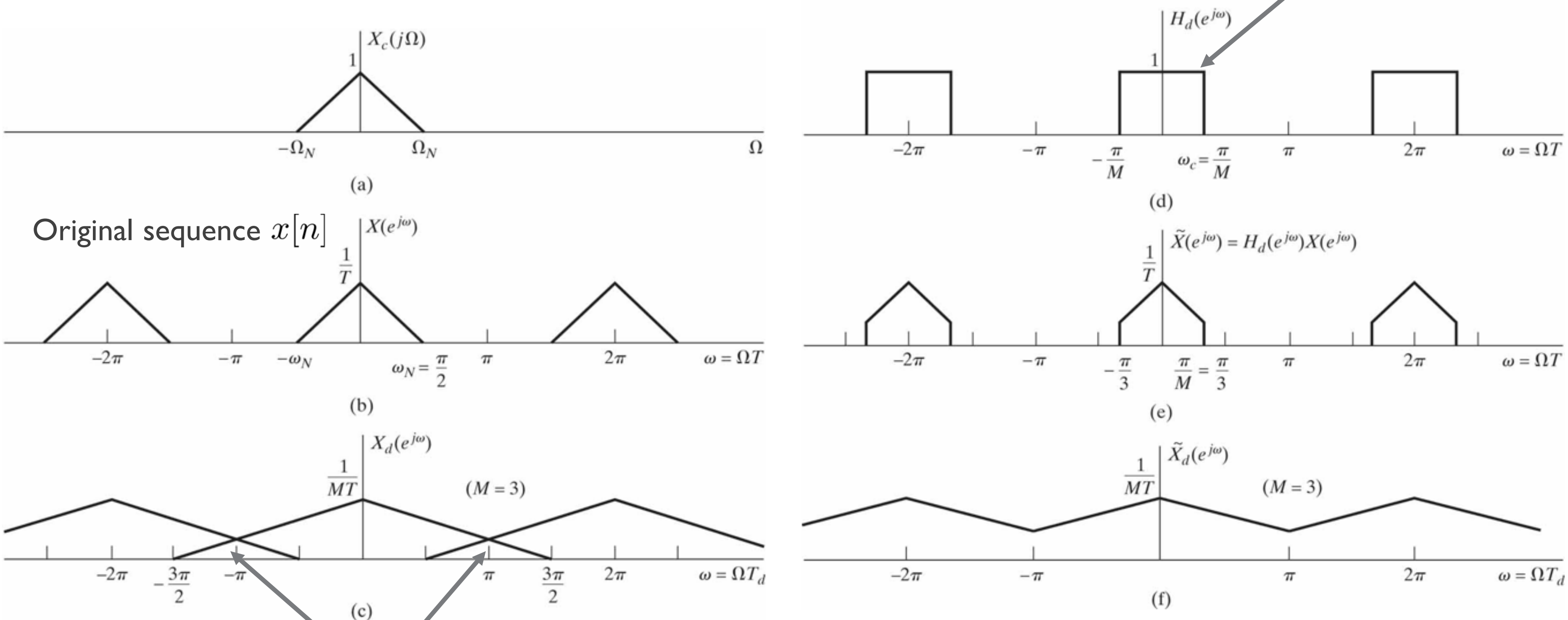




Example – with aliasing

Prefiltering to avoid aliasing

Original sequence $x[n]$



Aliasing occurs! To avoid aliasing, $\omega_N M \leq \pi$

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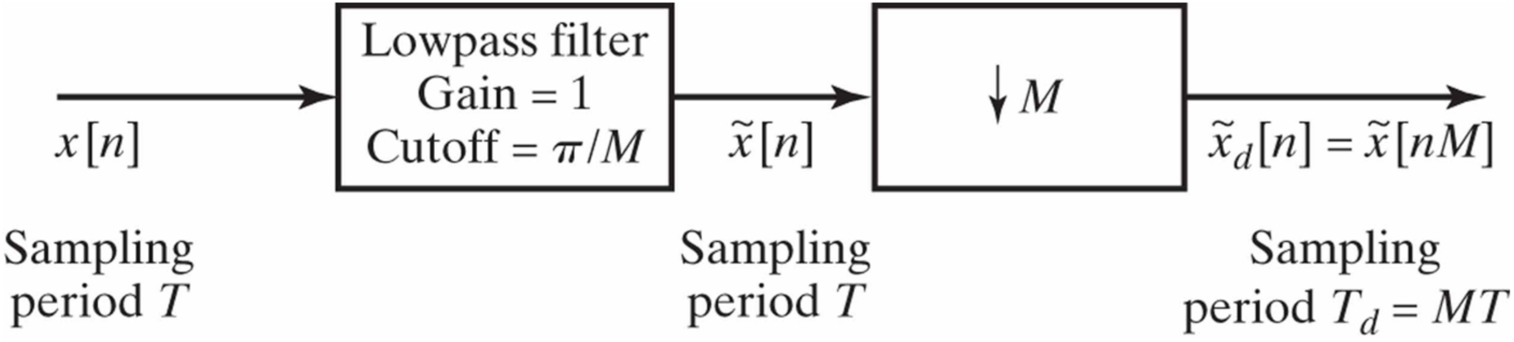
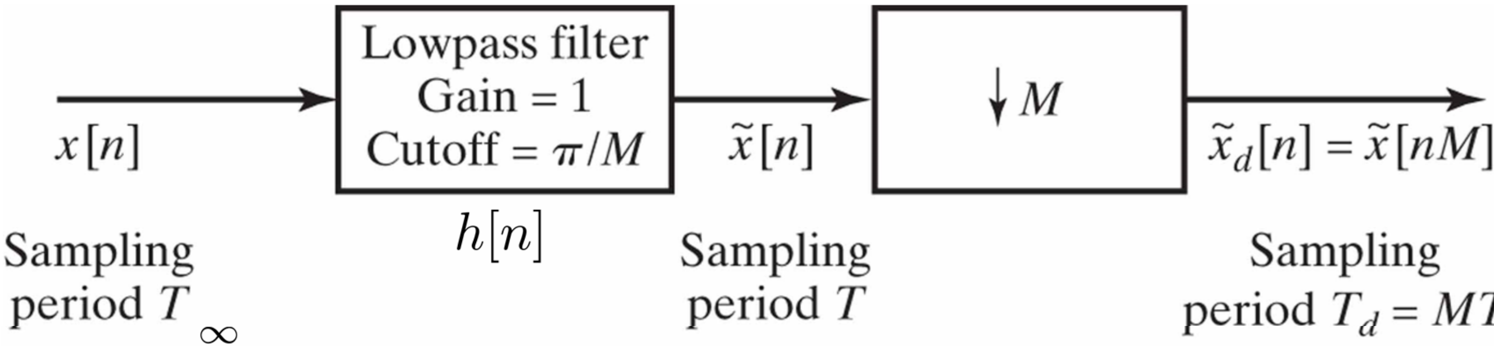
A general downsampling system



- Lowpass filter to avoid aliasing
- The system also called "decimator" (in general, "downsampling")

Efficient implementation of downsampling



- $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$\tilde{x}_d[n] = \sum_{k=-\infty}^{\infty} h[k]x[Mn-k] = \sum_{\ell=0}^{M-1} \sum_{k'=-\infty}^{\infty} h[k'M+\ell]x[Mn-(k'M+\ell)]$$

$$= \sum_{\ell=0}^{M-1} \sum_{k=-\infty}^{\infty} h[kM+\ell]x[M(n-k)-\ell] = \sum_{\ell=0}^{M-1} h[Mn+\ell] * x[Mn-\ell]$$

Block diagram representation