# Digital Signal Processing

**POSTECH**

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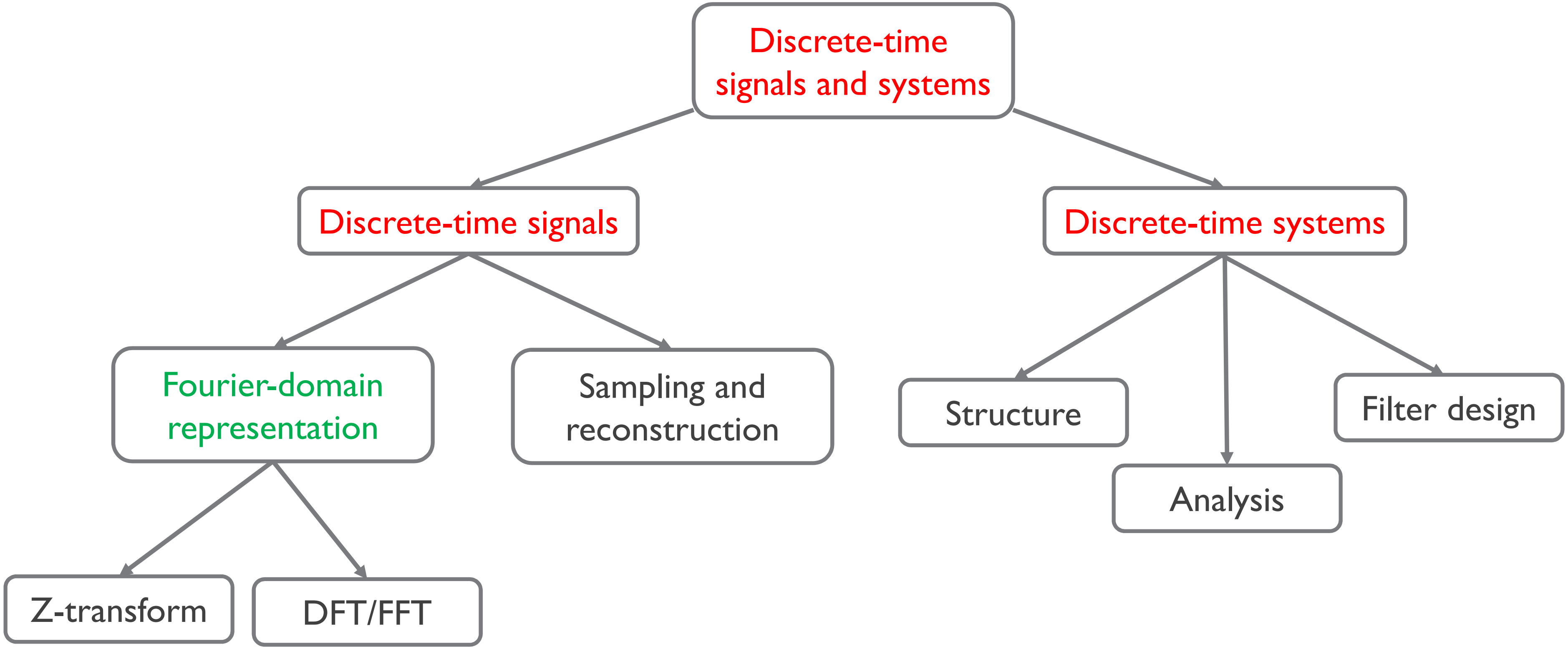
# Textbook HW password

- ???

# Course at glance



Discrete-time signals and systems

Discrete-time signals

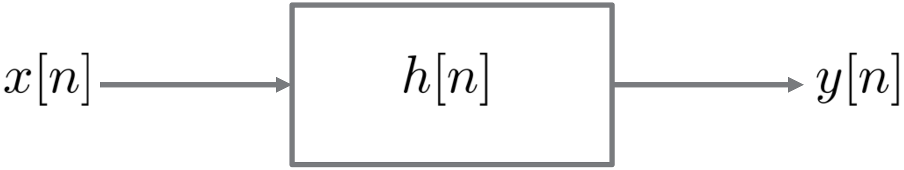
- Fourier-domain representation
  - Z-transform
  - DFT/FFT
- Sampling and reconstruction

Discrete-time systems

- Structure
- Analysis
- Filter design

# Frequency-domain description

- Consider LTI systems



- If x[n] is a complex exponential (or sinusoid)

⇒ Then y[n] is also a complex exponential (or sinusoid) with
1) Same frequency as x[n]
2) Amplitude and phase determined by the system h[n]

- In other words, complex exponential and sinusoid are eigenfunctions of LTI systems

# Eigenfunctions

- Consider complex exponential input $x[n] = e^{j\omega n}, \quad -\infty < n < \infty$

$$y[n] = T\{x[n]\} = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{j\omega n} = H(e^{j\omega})x[n]$$

where $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

- In other words, for LTI systems, if $x[n] = e^{j\omega n}, \quad -\infty < n < \infty$

$$y[n] = \underset{\text{Eigenvalue}}{\underset{\swarrow}{H(e^{j\omega})}}\underset{\text{Eigenfunction}}{\underset{\searrow}{e^{j\omega n}}}$$

$$\underset{\text{Linear operator}}{\underset{\swarrow}{A}}f = \lambda f$$

# Frequency response

- Eigenvalue $H(e^{j\omega})$ is called as frequency response of LTI system

- Mathematical definition

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

# Frequency response of moving average system

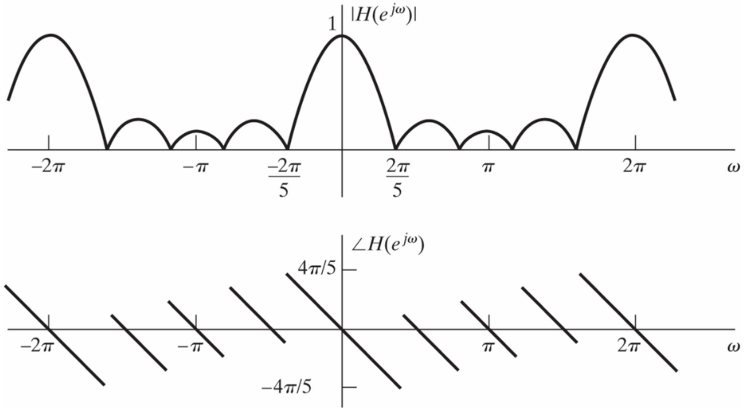
- ◆ Example 2.16 in the textbook
- ◆ General frequency response

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}$$

- ◆ If $M_1 = 0$,

$$H(e^{j\omega}) = \frac{1}{M_2 + 1} \sum_{n=0}^{M_2} e^{-j\omega n} = \frac{1}{M_2 + 1} \frac{\sin[\omega(M_2+1)/2]}{\sin \omega/2} e^{-j\omega M_2/2}$$



$M_1 = 0$

$M_2 = 4$

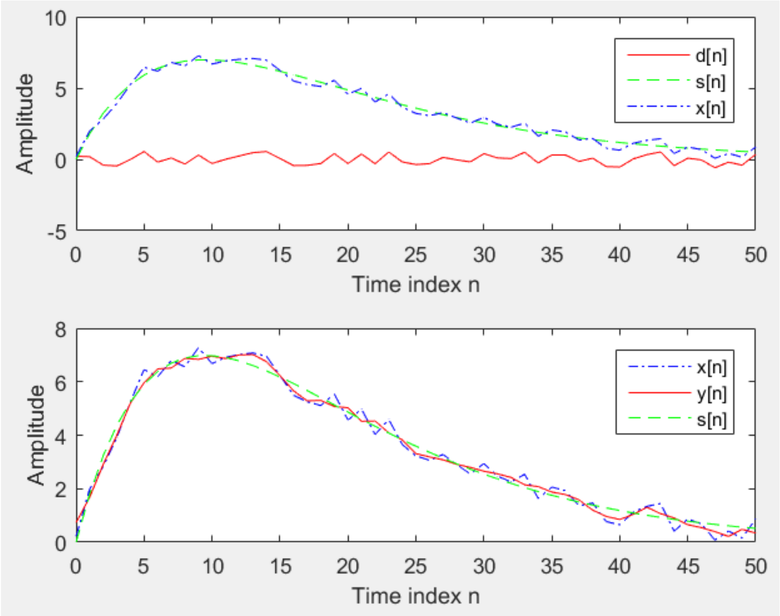
# Moving average

```
% Signal smoothing (moving average)

R=51;
d=1.2*(rand(1,R)-0.5);  % Generate random noise
m=0:R-1;
s=2*m.*(0.9.^m);        % Generate uncorrupted signal
x=s+d;                  % Generate noise corrupted signal

subplot(2,1,1);plot(m,d,'r-',m,s,'g--',m,x,'b-.');
xlabel('Time index n');ylabel('Amplitude');
legend('d[n]','s[n]','x[n]');

x1=[0 0 x];x2=[0 x 0];x3=[x 0 0];
y=(x1+x2+x3)/3;
subplot(2,1,2);plot(m,y(2:R+1),'r-',m,s,'g--');
legend('y[n]','s[n]');xlabel('Time index n');ylabel('Amplitude');
```



# Issues regarding practical signal

- In theory, we assume x[n] exists for $-\infty < n < \infty$
- This is rarely the case in real signals
  - Example: $x[n] = e^{j\omega n}u[n]$
- For causal LTI system

Is this obvious?

$$y[n] = \begin{cases} 0, & n < 0 \\ \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right) e^{j\omega n} & n \geq 0 \end{cases}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$

$$= H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$

Steady-state response $y_{ss}[n]$ Transient response $y_t[n]$

# Steady-state / transient responses

- For FIR system with $h[n] = 0$ except $0 \leq n \leq M$

$$y_t[n] = 0 \text{ for } n > M - 1$$

$$y[n] = y_{ss}[n] = H(e^{j\omega})e^{j\omega n} \quad \text{for } n > M - 1$$

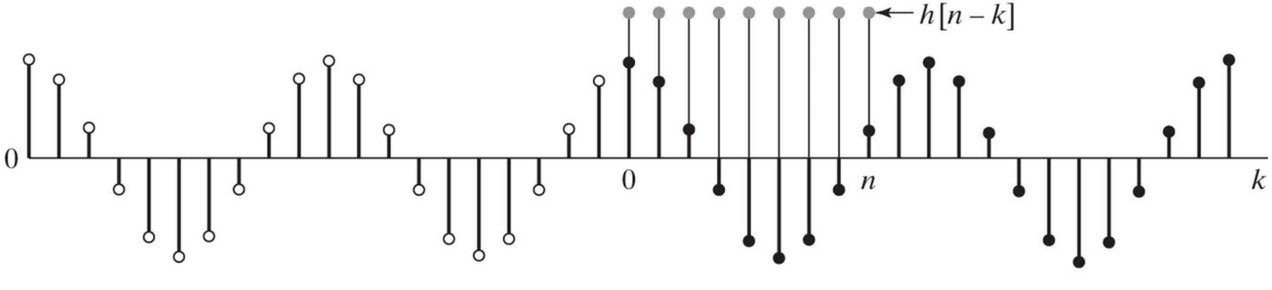
- For IIR system, transient period never ends, but…

$$|y_t[n]| = \left| \left( \sum_{k=n+1}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n} \right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]|$$

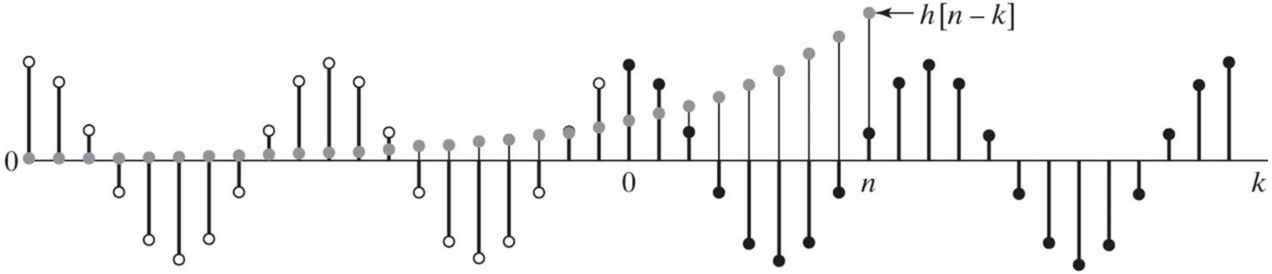
If the system is stable, i.e., $\sum_{k=0}^{\infty} |h[k]| < \infty$, $|y_t[n]|$ becomes increasingly smaller as $n \to \infty$

➡ Can rely on theoretical analysis assuming x[n] exists for $-\infty < n < \infty$

# Steady-state / transient responses



FIR case ➔ Only steady-state response remains after n



Stable IIR case ➔ Transient response becomes negligible as n increases

# Fourier Transform Representations

# Why frequency response important?

- A broad class of signals can be represented as $x[n] = \sum_k \alpha_k e^{j\omega_k n}$
  - Example: $A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$
- The output of LTI system is

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

- If we know $H(e^{j\omega_k})$ for all k, the output of the LTI system can be easily computed

# Frequency-domain representation

- Fourier transform pair
  - Many sequences can be represented as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ ➡ Inverse Fourier transform

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ ➡ (Discrete-time) Fourier transform

  - Note that x[n] is represented as a superposition of infinitesimally small complex sinusoids

$$\frac{1}{2\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

with $\omega$ ranging over an interval of length $2\pi$

# Meaning of Fourier transform

- Recall $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
- It computes how much of each frequency component $X(e^{j\omega})$ is required to represent

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Generally, Fourier transform is complex-valued function of $\omega$

$$\begin{aligned} X(e^{j\omega}) &= X_R(e^{j\omega}) + jX_I(e^{j\omega}) \\ &= |X(e^{j\omega})|e^{j\angle X(e^{j\omega})} \end{aligned}$$

Phase not uniquely specified

# Frequency response of LTI system

- Frequency response of LTI system

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

  is Fourier transform of impulse response h[n]

- Inverse Fourier transform gives

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

- Equivalence relation between the Fourier series representation of continuous-variable periodic functions and the Fourier transform representation of discrete-time signals
  - Read textbook 77p

# Sufficient condition for Fourier transform

- Fourier transform should be finite to exist

$$|X(e^{j\omega})| < \infty \quad \text{for all } \omega$$

- **Sufficient condition** for the convergence of $X(e^{j\omega})$

Not necessary

$$
\begin{aligned}
|X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty
\end{aligned}
$$

➡ All absolutely summable sequences have Fourier transform

# Not absolutely summable sequences

- Consider square summable $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

- For such sequences, mean-square convergence exists

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0$$

where

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

# Ideal lowpass filter example

- Frequency response of ideal lowpass filter

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

- Impulse response of $H_{\mathrm{lp}}(e^{j\omega})$

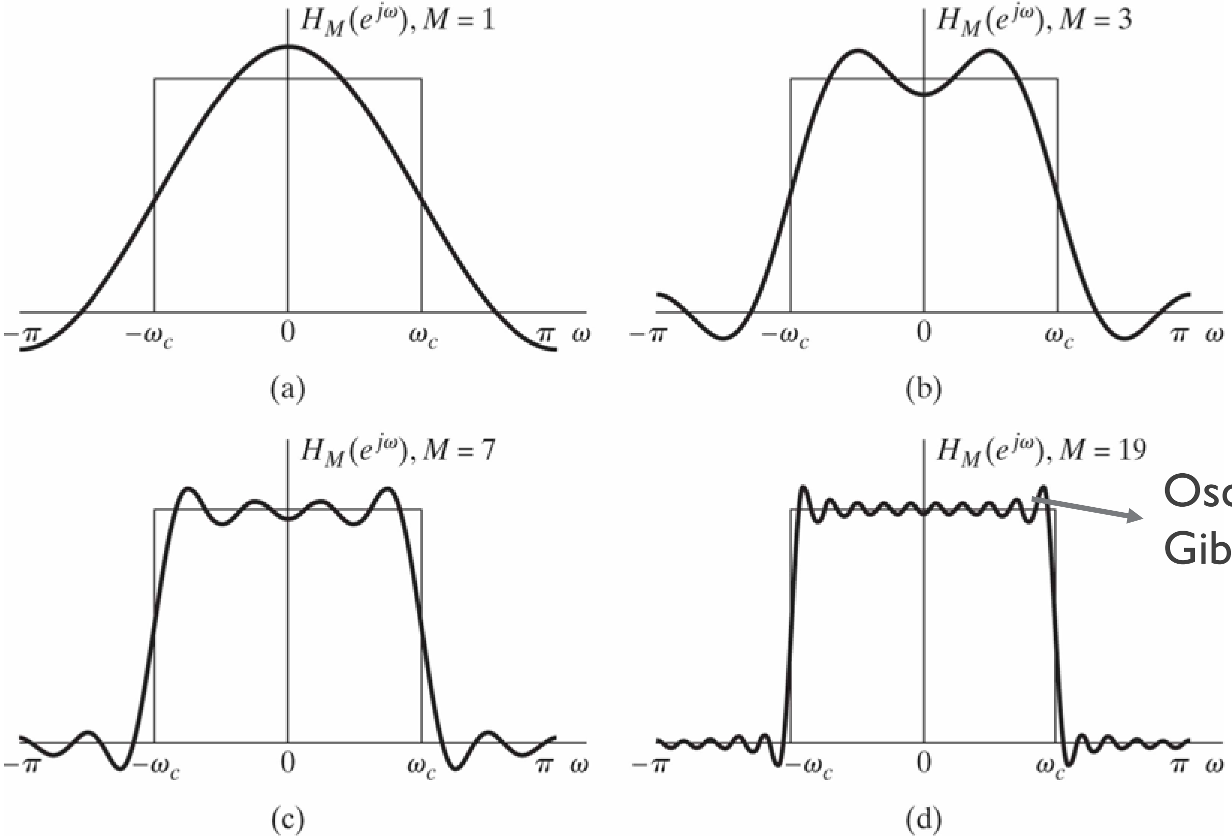
$$h_{\mathrm{lp}}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

➡ Noncausal, not absolutely summable

- Define $H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$ and check

$$\lim_{M\to\infty}\int_{-\pi}^{\pi} |H(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega$$

# Ideal lowpass filter example



(a) (b) (c) (d)

Oscillation called Gibbs phenomenon

# Sequences neither absolutely nor square summable

- Constant sequence: x[n]=1 for all n

Impulse train with period $2\pi$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

- Complex exponential sequence: $x[n] = e^{j\omega_0 n}$

Shifted impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r)$$

- Generalization of exponential sequence $x[n] = \sum_k a_k e^{j\omega_k n}$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

# Symmetry properties of Fourier Transform

# Symmetry properties of sequences

- Symmetry of complex sequence
  - Conjugate-symmetric: $x_e[n] = x_e^*[-n]$
  - Conjugate-asymmetric: $x_o[n] = -x_o^*[-n]$

- Any complex signal can be represented as $x[n] = x_e[n] + x_o[n]$ where

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]$$

# Symmetry properties of Fourier transform

- Similar to complex sequence,

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

where

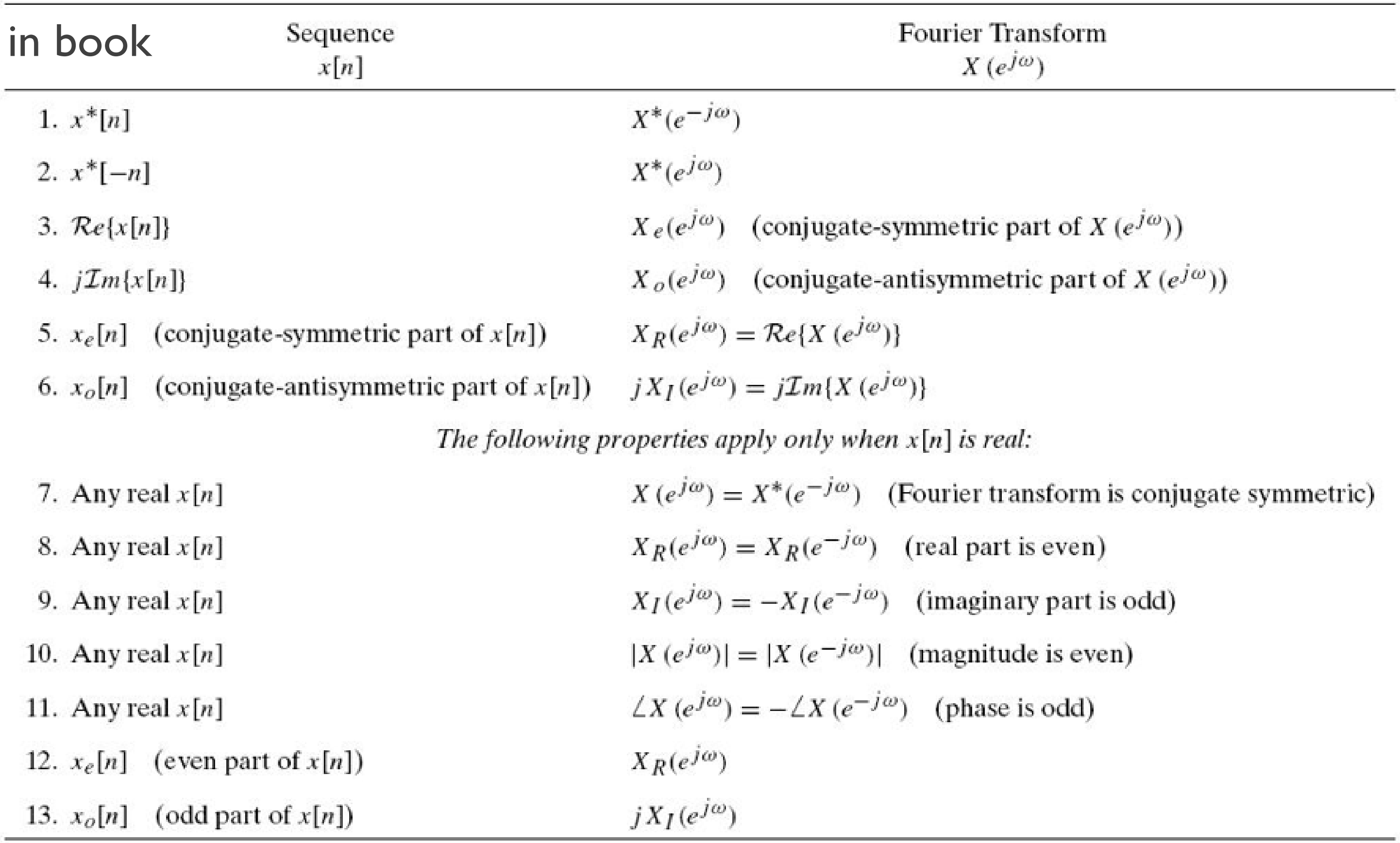
$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] = X_e^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})] = -X_o^*(e^{-j\omega})$$

- Symmetric properties can simplify solutions to problems

# Symmetry properties of Fourier transform

Table 2.1 in book



| Sequence $x[n]$ | Fourier Transform $X(e^{j\omega})$ |
|---|---|
| 1. $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 2. $x^*[-n]$ | $X^*(e^{j\omega})$ |
| 3. $\mathcal{Re}\{x[n]\}$ | $X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$) |
| 4. $j\mathcal{Im}\{x[n]\}$ | $X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$) |
| 5. $x_e[n]$ (conjugate-symmetric part of $x[n]$) | $X_R(e^{j\omega}) = \mathcal{Re}\{X(e^{j\omega})\}$ |
| 6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$) | $jX_I(e^{j\omega}) = j\mathcal{Im}\{X(e^{j\omega})\}$ |
| *The following properties apply only when $x[n]$ is real:* | |
| 7. Any real $x[n]$ | $X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric) |
| 8. Any real $x[n]$ | $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even) |
| 9. Any real $x[n]$ | $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd) |
| 10. Any real $x[n]$ | $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even) |
| 11. Any real $x[n]$ | $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd) |
| 12. $x_e[n]$ (even part of $x[n]$) | $X_R(e^{j\omega})$ |
| 13. $x_o[n]$ (odd part of $x[n]$) | $jX_I(e^{j\omega})$ |

# Simplify notations

- Define some operations

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} \{X(e^{j\omega})\}$$

# Fourier transform theorems

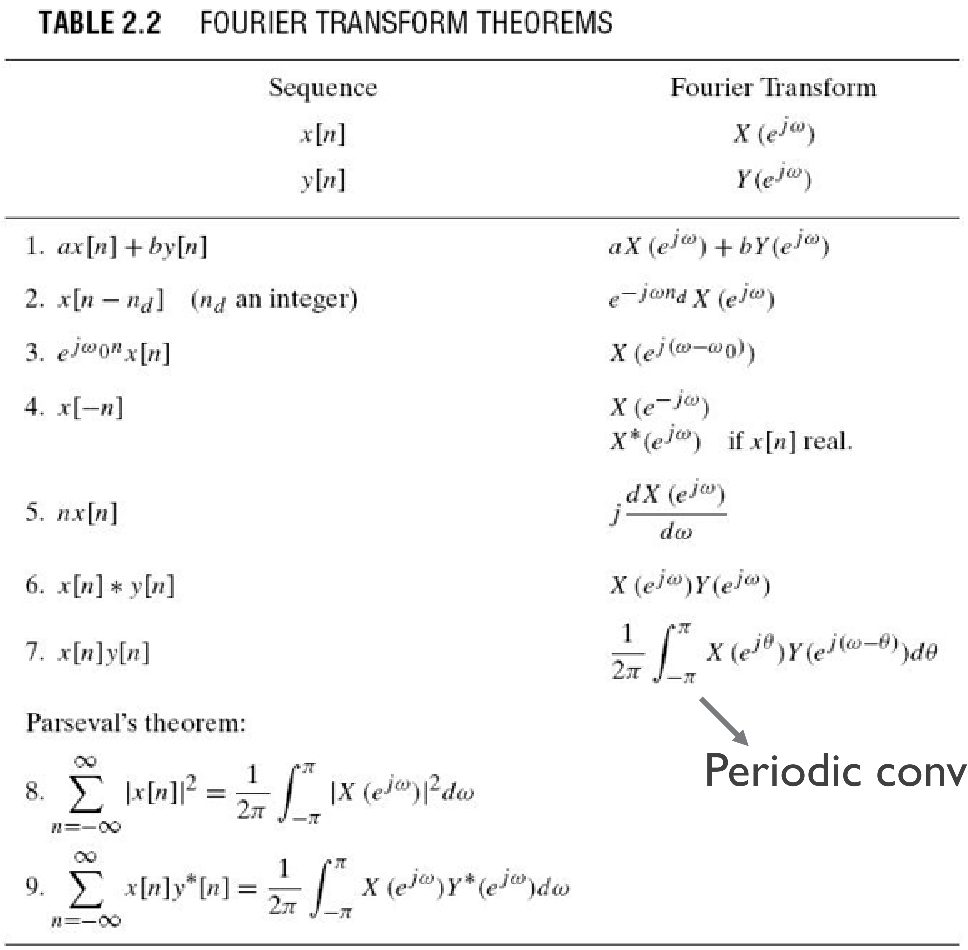


TABLE 2.2 FOURIER TRANSFORM THEOREMS

| Sequence | Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $y[n]$ | $Y(e^{j\omega})$ |
| 1. $ax[n]+by[n]$ | $aX(e^{j\omega})+bY(e^{j\omega})$ |
| 2. $x[n-n_d]$ ($n_d$ an integer) | $e^{-j\omega n_d}X(e^{j\omega})$ |
| 3. $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. $x[-n]$ | $X(e^{-j\omega})$<br>$X^*(e^{j\omega})$ if $x[n]$ real. |
| 5. $nx[n]$ | $j\dfrac{dX(e^{j\omega})}{d\omega}$ |
| 6. $x[n]*y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 7. $x[n]y[n]$ | $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| Parseval's theorem: | |
| 8. $\displaystyle\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$ | |
| 9. $\displaystyle\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$ | |

Periodic convolution

# Useful Fourier transform pairs

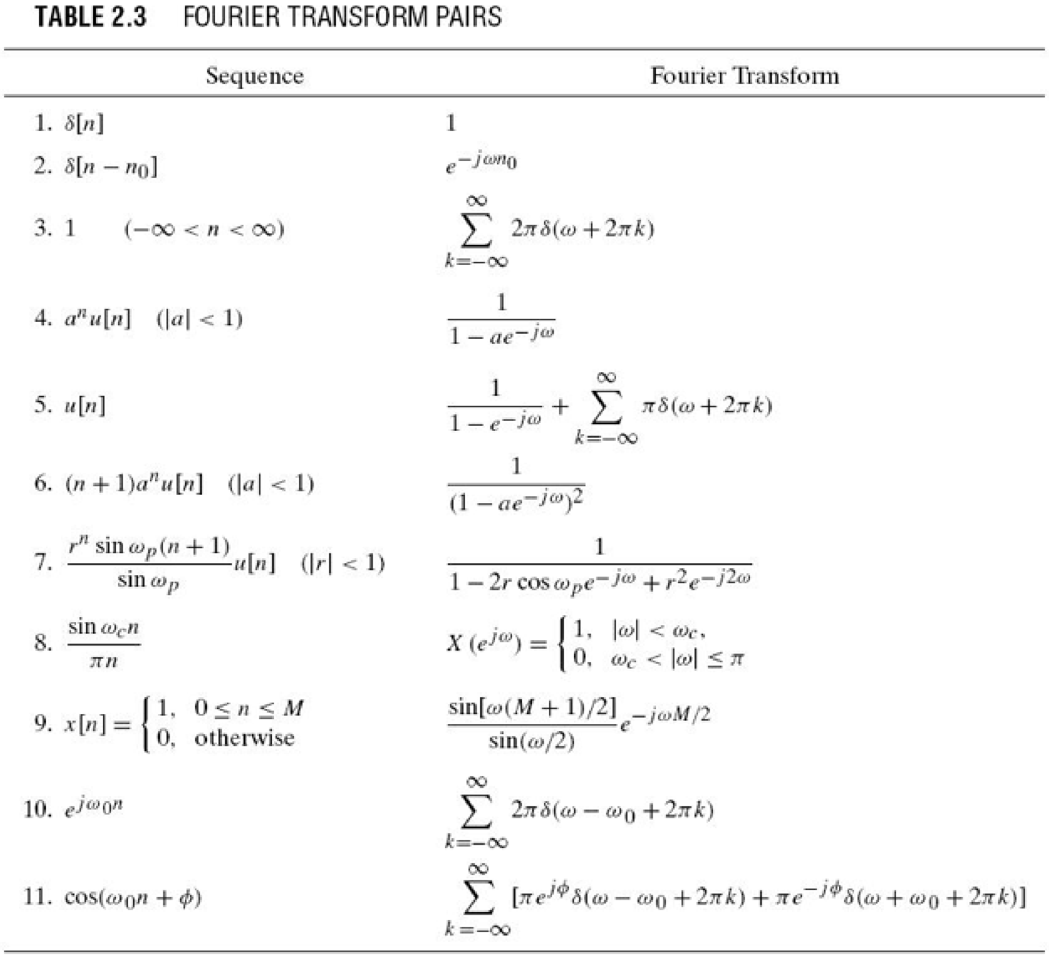


**TABLE 2.3** FOURIER TRANSFORM PAIRS

| Sequence | Fourier Transform |
|---|---|
| 1. $\delta[n]$ | $1$ |
| 2. $\delta[n-n_0]$ | $e^{-j\omega n_0}$ |
| 3. $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$ |
| 4. $a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| 5. $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$ |
| 6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| 7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$ | $\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$ |
| 8. $\frac{\sin\omega_c n}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$ |
| 9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$ |
| 10. $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$ |
| 11. $\cos(\omega_0 n+\phi)$ | $\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$ |

# Using Fourier transform theorems and pairs

- Using the tables, we can
  - Compute Fourier transform of sequences without complex computations
  - Compute inverse Fourier transform
  - Example: $\mathcal{F}\{a^n u[n-5]\} = ?$

$$x_1[n] = a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1 - ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}$$

# Another example

- Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$

- To get the impulse response, set $x[n] = \delta[n]$, which gives $y[n] = h[n]$

$$h[n] - \frac{1}{2}h[n-1] = \delta[n] - \frac{1}{4}\delta[n-1]$$

- Fourier transform gives

$$H(e^{j\omega}) - \frac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

# Another example

- Decompose $H(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}} - \dfrac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$

- By using the table

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}, \qquad -\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1] \overset{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

- h[n] becomes

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- Read Examples 2.23-2.24 in the textbook

# Auto/cross-correlations of Deterministic Signals

# Definitions

- Autocorrelation $c_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x^*[n-\ell]$

- Cross-correlation $c_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y^*[n-\ell]$

Conjugate for complex sequences

- Relation to convolution: missing initial fold step

$$c_{xy}[\ell] = x[\ell] * y^*[-\ell]$$
$$c_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

# Pseudo-noise (PN) sequence

$$\{1,1,1,-1,-1,1,-1\}$$

$$\{1,1,1,-1,-1,1,-1\} \longrightarrow$$

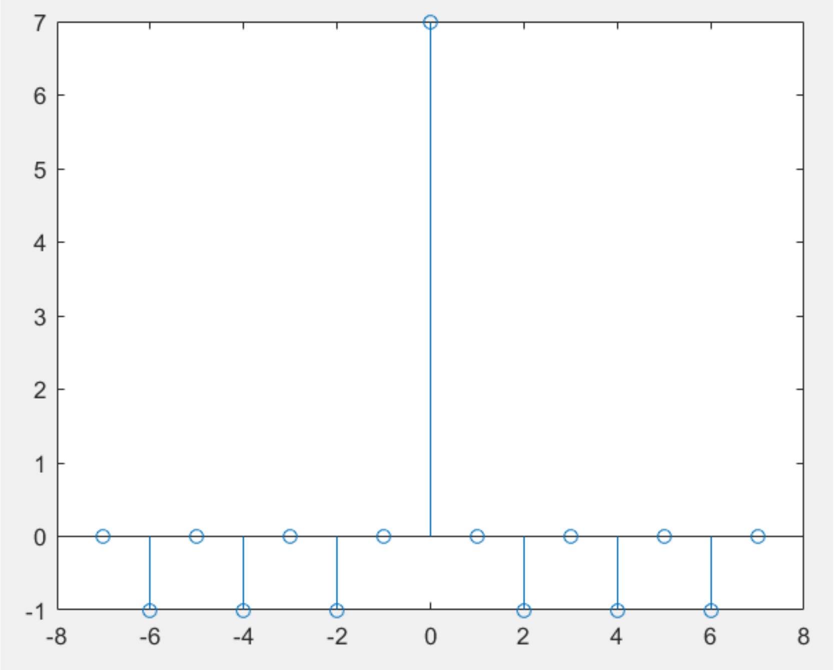
- Let $x[n] = \{1, 1, 1, -1, -1, 1, -1\}$

  $\nearrow$ $n = 0$

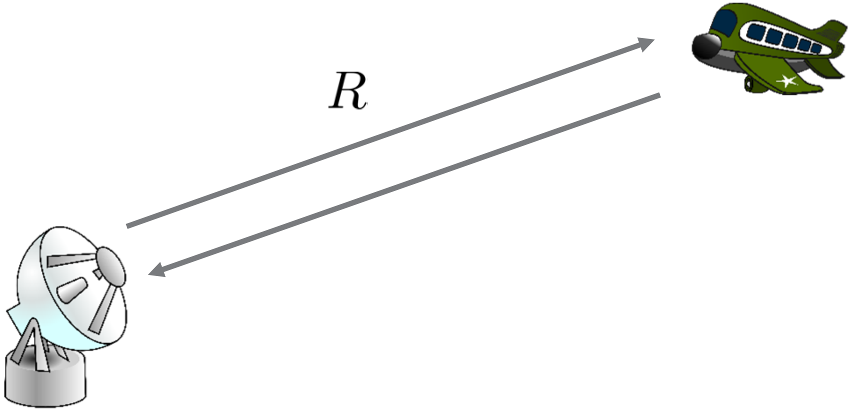
- Autocorrelation

$$c_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x^*[n-\ell]$$

- Example of Barker code
- Sharp peak at $\ell = 0$
- Time-delay estimation in radar



# Radar example



- Transmit pulse $S_a(t)$
- Received "echo" after reflection off object

$$y_a(t) = \Gamma S_a(t - \tau_d) + w_a(t)$$

Unknown amplitude

Round-trip time-delay

Noise

$$\tau_d = \frac{2R}{c}$$

# Radar example

- Sampled version $y[n] = \Gamma S_a(nT_s - \tau_d) + w[n]$
- Assume sampling high enough: $\tau_d = DT_s, \text{ where } D \text{ is integer}$

$$S_a(nT_s - DT_s) = S_a((n-D)T_s) = s[n-D], \text{ where } s[n] = S_a(nT_s)$$

- Discrete-time (DT) model

$$y[n] = \Gamma s[n-D] + w[n] = s[n] * \Gamma\delta[n-D] + w[n]$$

  - Without noise, it can be modeled as an LTI system

$$x[n] \longrightarrow \boxed{h[n] = \Gamma\delta[n-D]} \longrightarrow y[n]$$

- Use cross-correlation to estimate D ➔ $R = \dfrac{cDT_s}{2}$

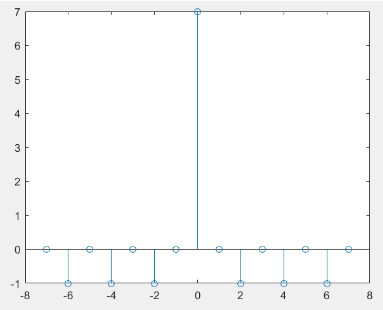
# Radar example

◆
$$
\begin{aligned}
c_{ys}[\ell] &= \sum_{n=-\infty}^{\infty} y[n]s[n-\ell] \\
&= \sum_{n=-\infty}^{\infty} \left(\Gamma s[n-D] + w[n]\right) s[n-\ell] \\
&= \sum_{n=-\infty}^{\infty} \Gamma s[n-D]s[n-\ell] + \sum_{n=-\infty}^{\infty} w[n]s[n-\ell] \\
&= \Gamma \sum_{n'=-\infty}^{\infty} s[n']s[n'-(\ell-D)] + c_{ws}[\ell] \\
&= c_{ss}[\ell-D] + c_{ws}[\ell]
\end{aligned}
$$

Assume s[n] is a real signal, e.g., Barker code

$n' = n - D$

Peak at $\ell = D$



# Radar example: two-target case

- Assume noiseless case

$$
\begin{aligned}
y[n] &= \Gamma_1 s[n - D_1] + \Gamma_2 s[n - D_2] \\
&= s[n] * \{\Gamma_1 \delta[n - D_1] + \Gamma_2 \delta[n - D_2]\} \\
&= s[n] * h[n]
\end{aligned}
$$

- Cross-correlation between s[n] and y[n]

$$c_{ys}[\ell] = y[\ell] * s^*[-\ell] = c_{ss}[\ell] * h[\ell] = \Gamma_1 c_{ss}[\ell - D_1] + \Gamma_2 c_{ss}[\ell - D_2]$$

# Radar example: two-target case

- Desired autocorrelation sequence

$$c_{ys}[\ell] = \Gamma_1 c_{ss}[\ell - D_1] + \Gamma_2 c_{ss}[\ell - D_2]$$

Want to have an approximated Kronecker delta $\delta[n]$
to resolve closely-spaced targets

- Some undesired scenarios
  - Two autocorrelations can overlap ➔ yield only a single peak at some delay between $D_1$ and $D_2$
  - Weaker (smaller) target can be masked by the "sidelobes" of the stronger target

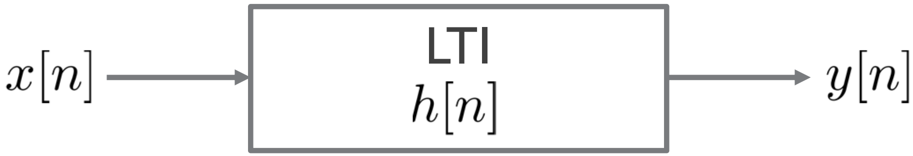
# Properties of autocorrelation sequences

- Three main properties of autocorrelation sequence

$$c_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

  - $c_{xx}[-\ell] = c_{xx}^*[\ell]$
  - $|c_{xx}[\ell]| \le c_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \text{energy}$
  - $\sum_{\ell=-\infty}^{\infty} c_{xx}[\ell] e^{-j\omega\ell} \ge 0, \text{ for all } \omega$

# Input/output relationships for LTI system



- $c_{yx}[\ell] = c_{xx}[\ell] * h[\ell]$
- $c_{yy}[\ell] = c_{xx}[\ell] * c_{hh}[\ell]$

Try to prove these

# MATLAB Programming

# DTFT computation - preliminary

- Use ‘freqz’ function

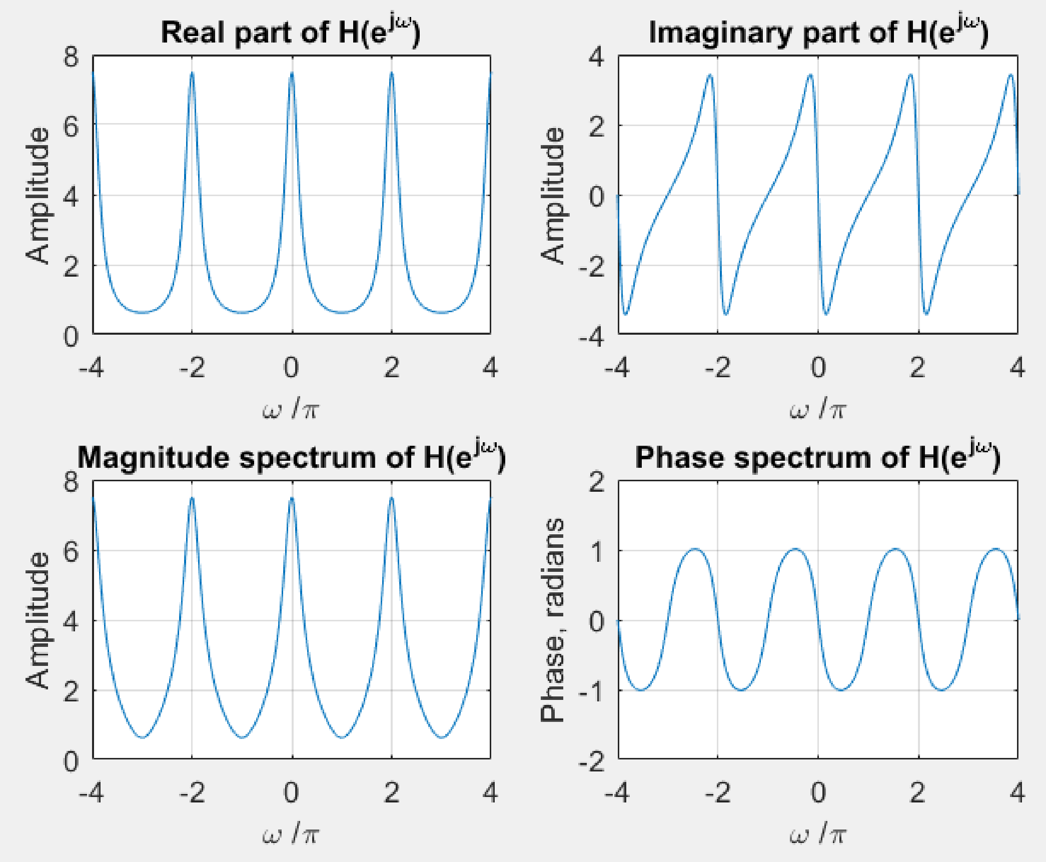
- Only can evaluate DTFT that is expressed as

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$$

# DTFT computation

```
num=[2 1];
den=[1 -0.6];
w=-4*pi:8*pi/511:4*pi;
h=freqz(num,den,w);

subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude spectrum of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase spectrum of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Phase, radians')
```

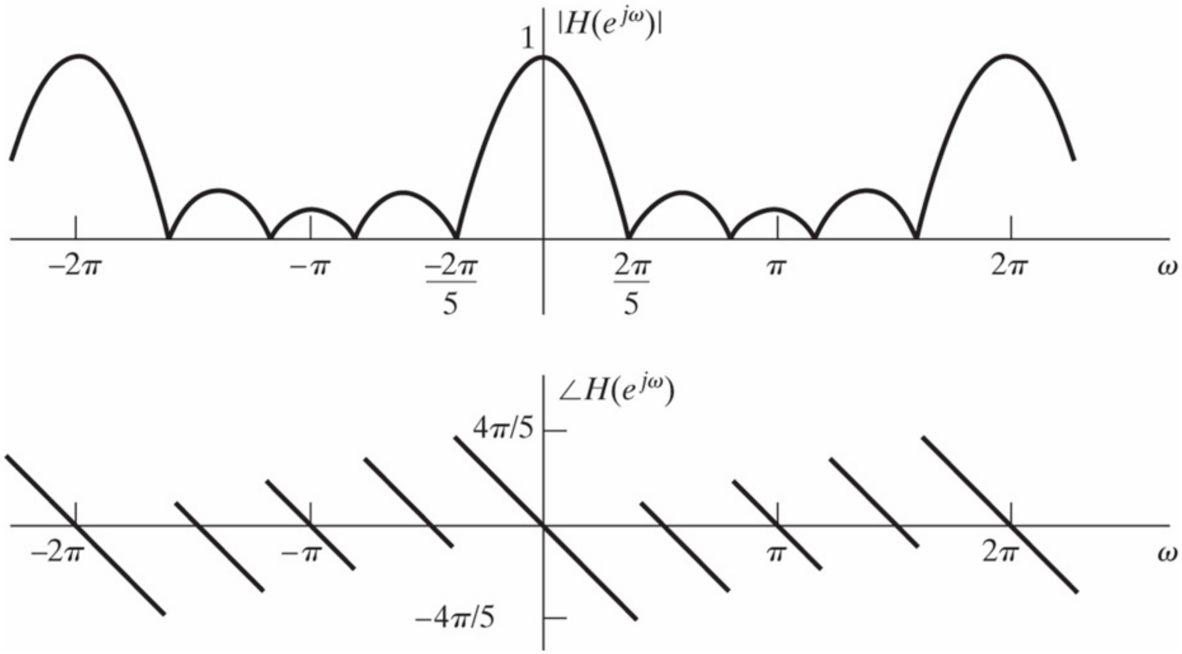


# Homework

- Problems in textbook: 2.36, 2.37, 2.43, 2.45, 2.58
- MATLAB problems
  - Submit by 10/2 (Tuesday) before the class
    - Make a document containing releveant figures and answers for questions
      - No hard copy required.
    - Send a zip file containing all m-files, plots, and the document.

# MATLAB problem 1

- Plot the graphs in Example 2.16 using equation (2.123) with $M_2 = 4$
- Use 'freqz' to plot the same graph



# MATLAB problem 2

- Time-shift property of DTFT is

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) \Rightarrow x[n-n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} X(e^{j\omega})$$

- Using 'freqz' function, write a MATLAB program to check time-shift property
  - $x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$ with $n_d = 5$
  - Plot magnitudes and phases of original and delayed sequences
  - Use 'unwrap' function to remove sudden jumps in phases
  - Interpret the phase plots to show time-shift property holds

# MATLAB problem 3

- Convolution property of DTFT is

$$x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- Using 'freqz' function, write a MATLAB program to check convolution property
  - Use
$$x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10]), y[n] = (3)^n (u[n] - u[n-7])$$
  - You should compare the Fourier transforms of $x[n] * y[n]$ and $X(e^{j\omega})Y(e^{j\omega})$
  - Plot both magnitudes and phases

# MATLAB problem 4 (require 6 plots total)

- Consider the radar example

$$y[n] = \Gamma s[n-D] + w[n] = s[n] * \Gamma\delta[n-D] + w[n]$$

- Define variables

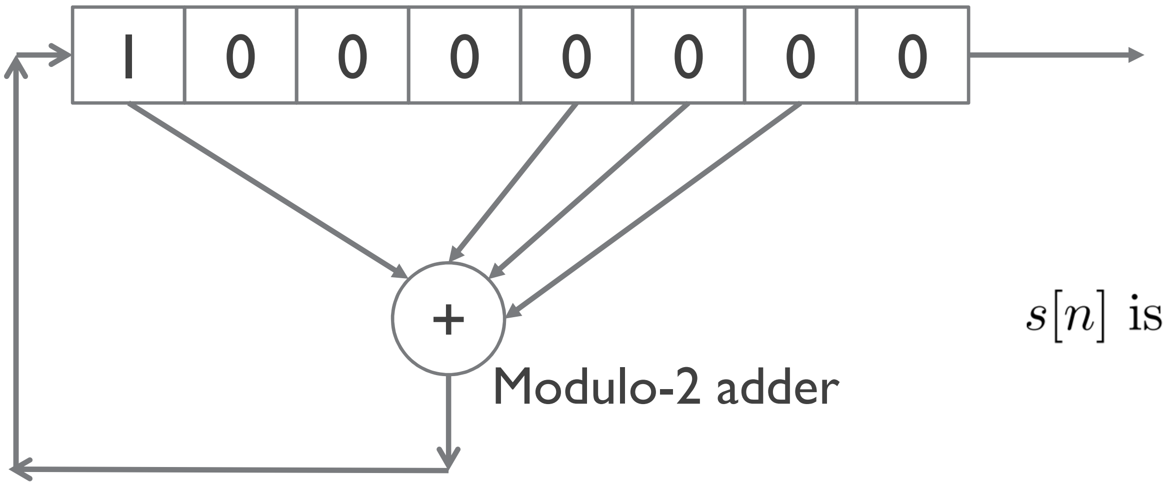
$s[n] = \{1,1,1,1,1,-1,-1,1,1,-1,1,-1,1\},\ \Gamma = 0.9,\ D = 20$

$w[n]:\ \text{Gaussian random sequence with zero mean and variance } \sigma^2 = 0.01$

- Write a program to plot y[n] for $0 \le n \le 199$
- Compute and plot the cross-correlation $c_{ys}[\ell]$ for $0 \le \ell \le 59$
  - Can we estimate D with this cross-correlation?
- Repeat all parts with $\sigma^2 = 0.1$ and $\sigma^2 = 1$
  - What is the role of $\sigma^2$ in finding D? Is it helpful or not?

# MATLAB problem 5 (require 6 plots total)

- Repeat "MATLAB problem 4" with s[n] obtained from the linear shift register



Output=s[n]

$0 \rightarrow -1$

$1 \rightarrow +1$

$s[n]$ is $2^8 - 1$-length sequence

- When plotting y[n], set $0 \leq n \leq 500$. For $c_{ys}[\ell]$, set $0 \leq \ell \leq 99$
- Is having a longer sequence s[n] beneficial to detect the target in radar?