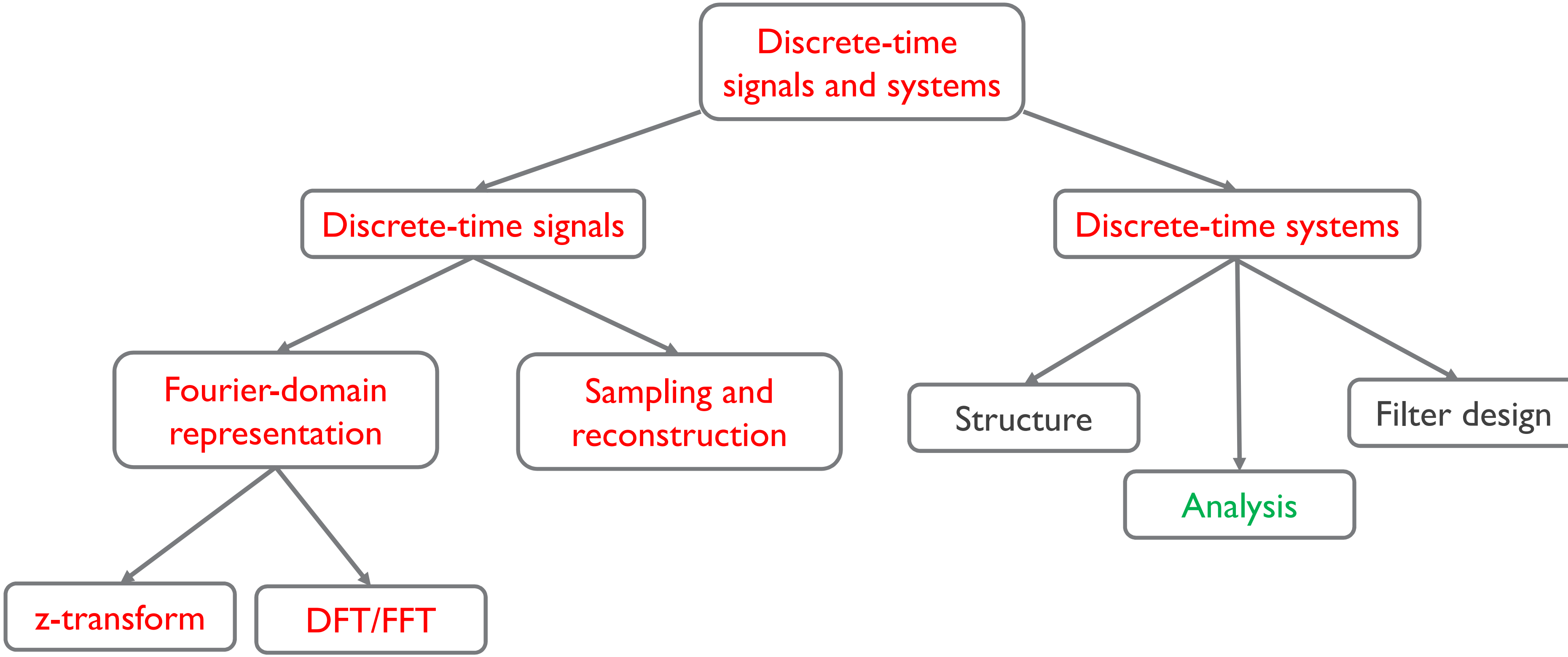
Digital Signal Processing

POSTECH
Department of Electrical Engineering
Junil Choi

Course at glance



LTI System Analysis

Magnitude and phase of output

- Fourier transform is in general a complex number

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$
$$= |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

- Magnitude and phase of the Fourier transforms of system input and output

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

$$|H(e^{j\omega})| : \text{ magnitude response or gain}$$
$$\angle H(e^{j\omega}) : \text{ phase response or phase shift}$$

Effect on magnitude and phase

- The effects

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

  may or may not be desirable

- For undesirable effects, often refer to the effects as magnitude and phase distortions

Phase of general complex numbers

- Phase is not uniquely defined
  - Period of 2π

- Denote the principal value of the phase of $H(e^{j\omega})$

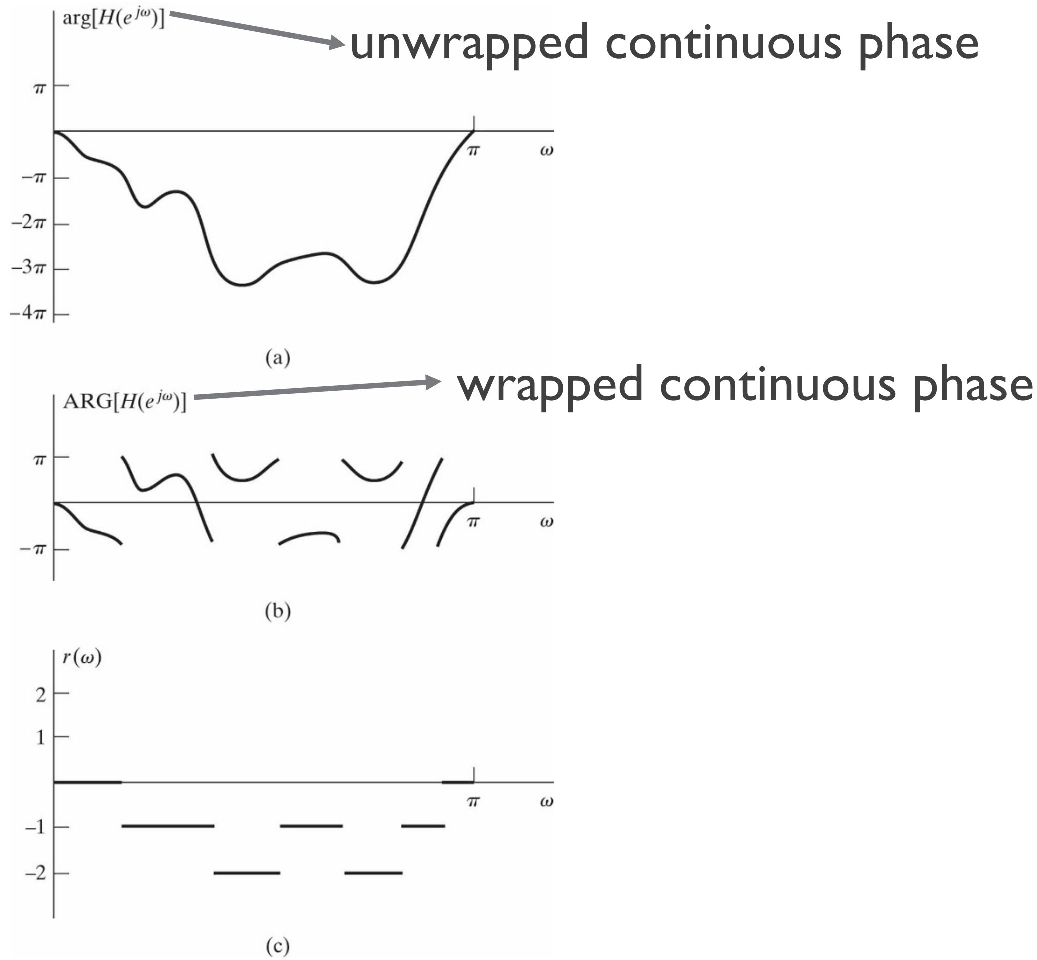
$$-\pi < \mathrm{ARG}[H(e^{j\omega})] \leq \pi$$

- General angle notation

$$\angle H(e^{j\omega}) = \arg[H(e^{j\omega})] = \mathrm{ARG}[H(e^{j\omega})] + 2\pi r(\omega)$$

Arbitrary integer that may depend on ω

Discontinuity of principal value of phase



Group delay

- The group delay is defined as

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

- Using $\mathrm{ARG}[H(e^{j\omega})]$ also possible
  - Need to take possible discontinuities into account

Example of ideal delay system

- Impulse response

$$h_{\mathrm{id}}[n] = \delta[n - n_d]$$

- Frequency response

$$H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega n_d}$$

- Magnitude and phase responses

$$|H_{\mathrm{id}}(e^{j\omega})| = 1$$

$$\angle H_{\mathrm{id}}(e^{j\omega}) = -\omega n_d, \quad |\omega| < \pi$$

Phase response linear with frequency

Delay distortion and phase distortion

- In general, phase distortion is nonlinear with frequency
- Delay distortion (i.e., linear phase distortion) is rather a mild phase distortion
  - Can be compensated easily
- When designing filters or other LTI systems, frequently accept a linear-phase response (while zero-phase response is ideal)

Narrowband input

- Assume $X(e^{j\omega})$ is a narrowband signal, i.e., $x[n] = s[n]\cos(\omega_0 n)$
  - $X(e^{j\omega})$ is nonzero only around $\omega = \omega_0$
- The effect of the phase of the system can be linearly approximated as

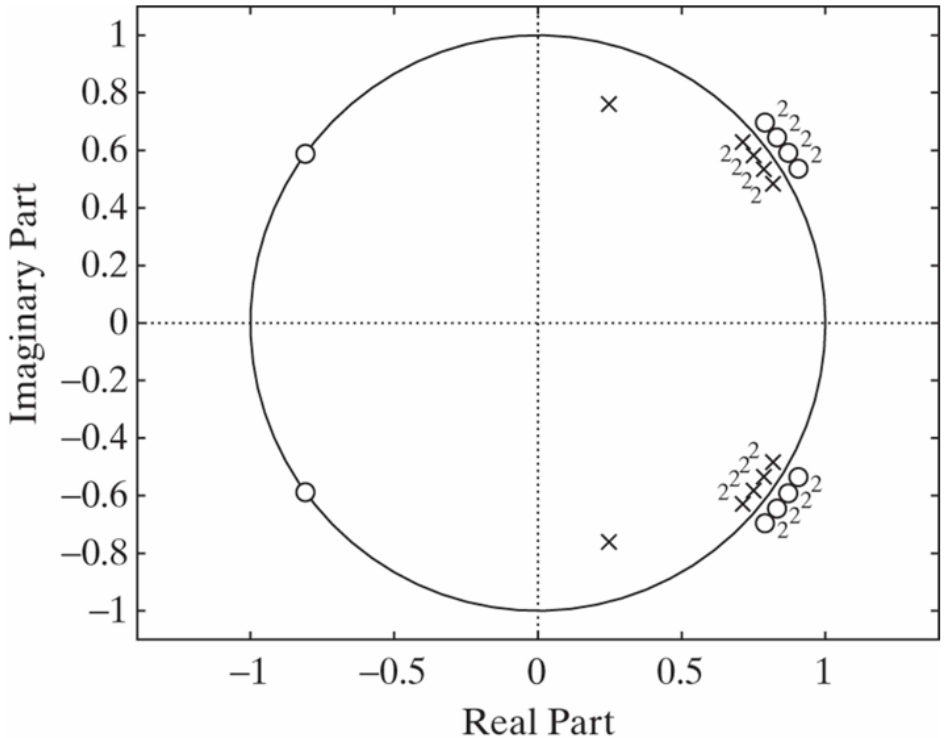
$$\arg[H(e^{j\omega})] \simeq \underbrace{-\phi_0}_{\text{Constant term}} \underbrace{- \omega n_d}_{\text{Group delay}}$$

- Consider wideband signal as a superposition of narrowband signals
  - Constant group delay with frequency ➔ each narrowband component will undergo identical delay
  - Nonconstant group delay ➔ result in time dispersion

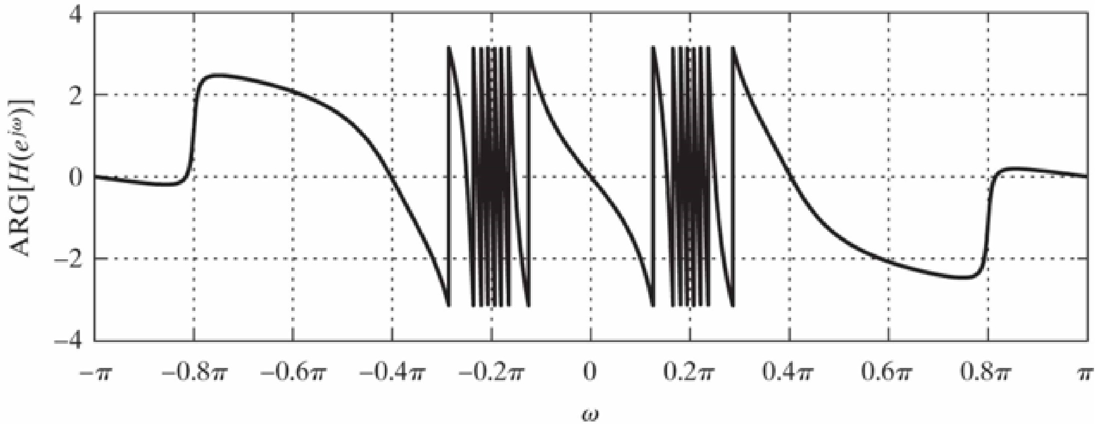
Effects of group delay and attenuation

- Consider the system with pole-zero plot

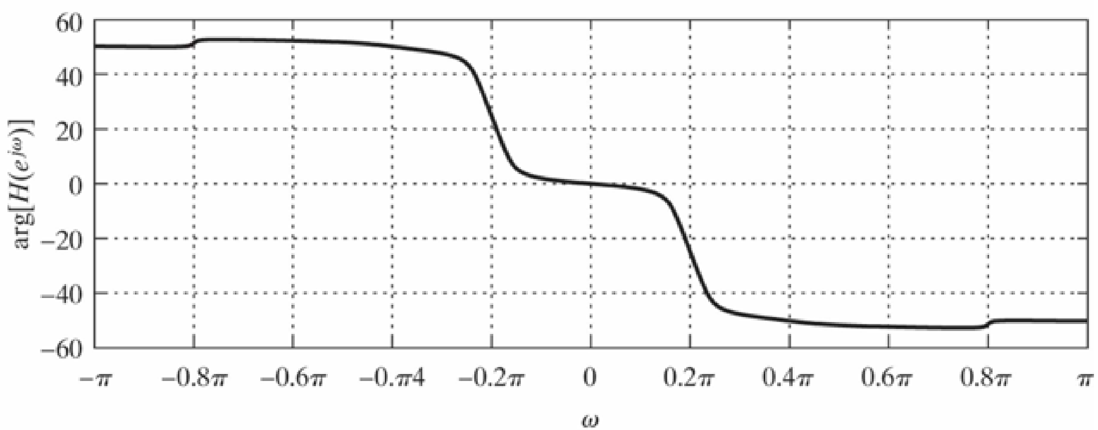
Figure 5.2 Pole-zero plot for the filter in the example of Section 5.1.2. (The number 2 indicates double-order poles and zeroes.)



Phase response

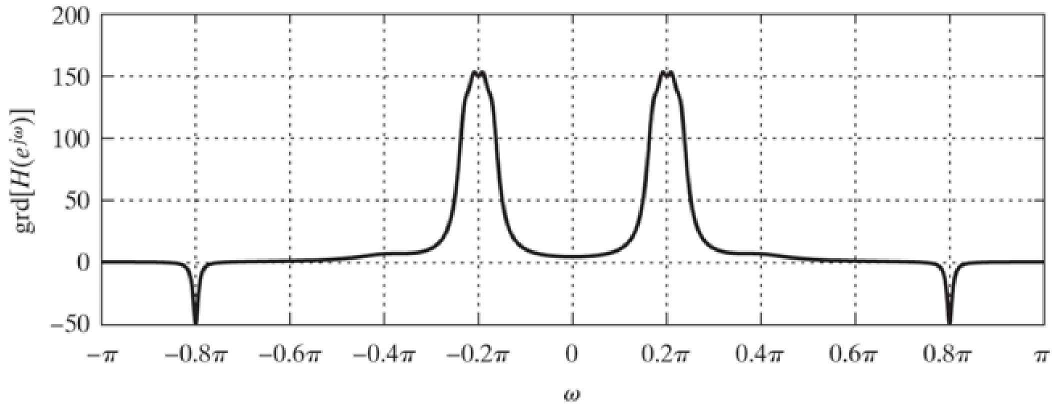


(a) Principle Value of Phase Response

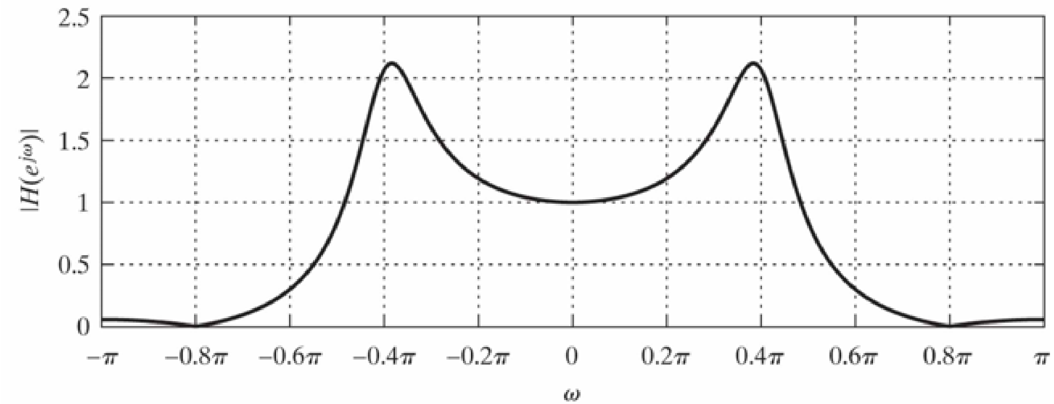


(b) Unwrapped Phase Response

Group delay and magnitude response

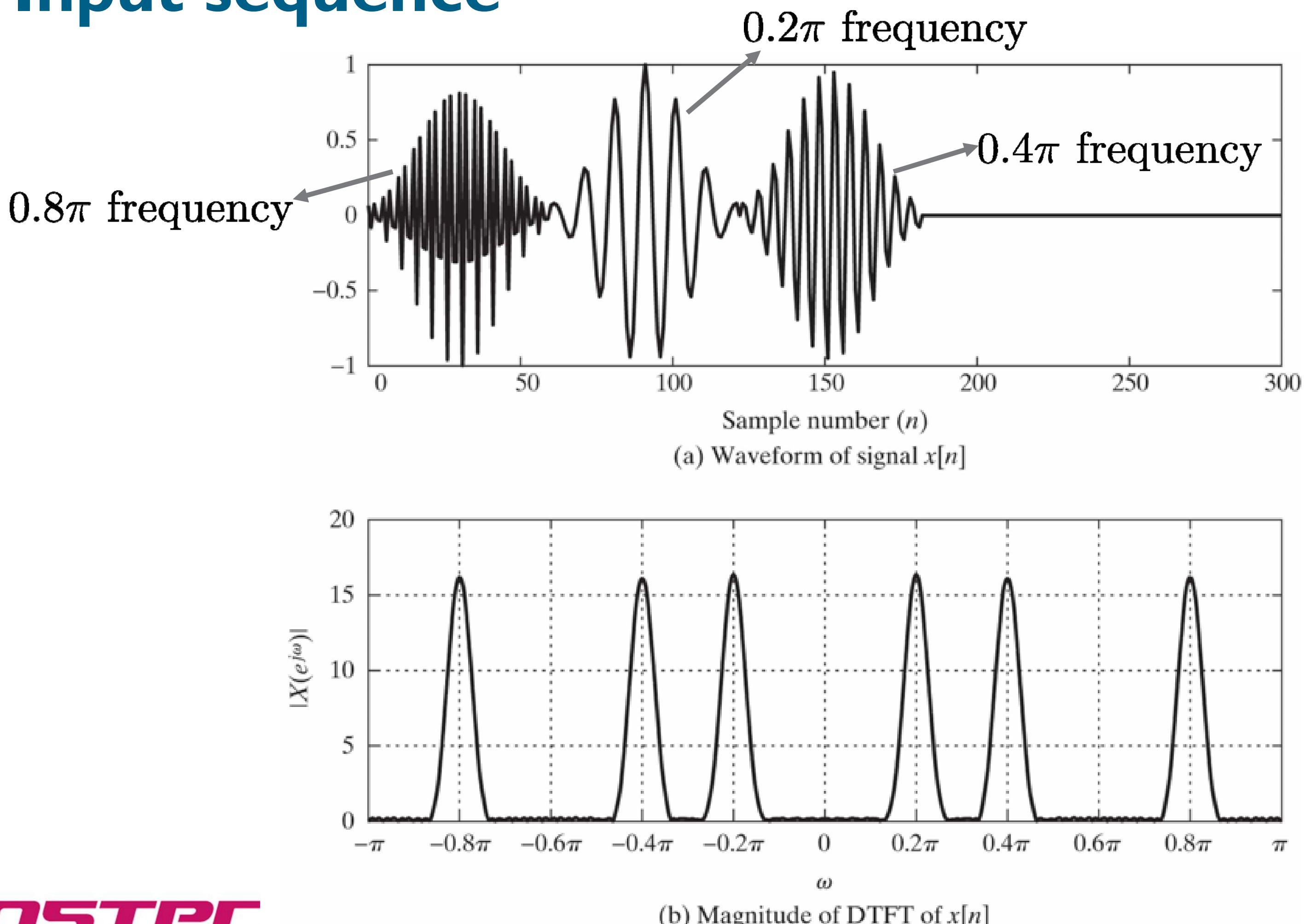


(a) Group delay of $H(z)$

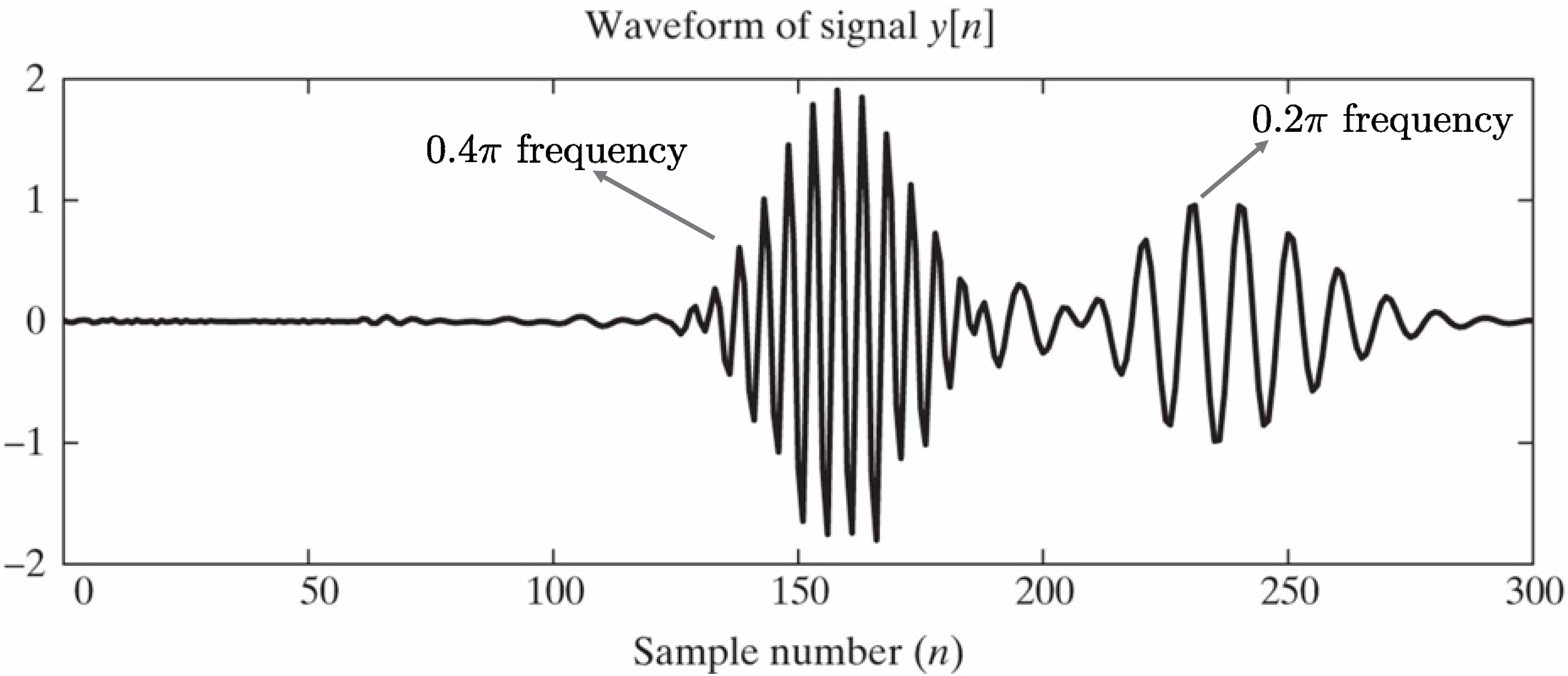


(b) Magnitude of Frequency Response

Input sequence



Output sequence



Review from Previous Lectures

z-transform for difference equations

- z-transform is particularly useful for LTE systems with difference equations

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right) x[n-k]$$

- Due to linearity and time-shift properties

$$Y(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) z^{-k}Y(z) + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right) z^{-k}X(z)$$

$$\Rightarrow Y(z) = \left(\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}\right) X(z)$$

$$H(z)!$$

Example

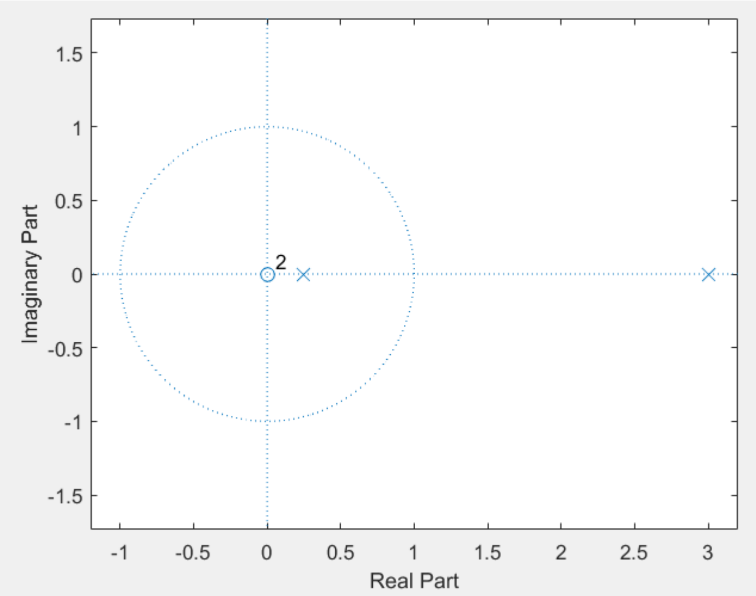
- Let $y[n] = \frac{13}{4}y[n-1] - \frac{3}{4}y[n-2] + x[n]$

- z-transform gives

$$Y(z) = \frac{13}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{13}{4}z^{-1} + \frac{3}{4}z^{-2}}$$

$$= \frac{z^2}{z^2 - \frac{13}{4}z + \frac{3}{4}} = \frac{z^2}{\left(z - \frac{1}{4}\right)(z-3)}$$



Example

- Using partial fraction expansion

$$H(z) = -\frac{1}{11}\frac{z}{z-\frac{1}{4}} + \frac{12}{11}\frac{z}{z-3}$$

- Three possibilities for ROC
  - $|z| < \frac{1}{4}$
  - $\frac{1}{4} < |z| < 3$
  - $|z| > 3$

Example

- If $|z| < \frac{1}{4}$

- Impulse response becomes

$$h[n] = \frac{1}{11}\left(\frac{1}{4}\right)^n u[-n-1] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Left-sided sequence

- BIBO stable? No! $\lim_{n \to -\infty} |h[n]| = \infty$

Example

- If $\frac{1}{4} < |z| < 3$

- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] - \frac{12}{11}(3)^n u[-n-1]$$

- Causal? No! Two-sided sequence

- BIBO stable? Yes! $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Example

- If $|z| > 3$

- Impulse response becomes

$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{11}(3)^n u[n]$$

- Causal? Yes!

- BIBO stable? No!

Stability and causality

- Stability requires ROC to include unit circle $|z| = 1$
  - Proof using triangle inequality $|a + b| \le |a| + |b|$

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h[n] z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}| \overset{|z|=1}{=} \underbrace{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{For BIBO stability}}$$

- Causality requires ROC to satisfy $|z| > |p_N|$

Largest pole

- If the system to be stable AND causal
  ➔ $|p_N| < 1$
  ➔ All poles must be located within unit circle

New Concepts

Inverse systems

- Definition: $H_i(z) = \dfrac{1}{H(z)}$
- Time-domain condition: $h[n] * h_i[n] = \delta[n]$
- Frequency response of inverse system (if it exists): $H_i(e^{j\omega}) = \dfrac{1}{H(e^{j\omega})}$
  - Not all systems have an inverse
    - Ideal lowpass filter does not have an inverse

Inverse with rational form

- Consider

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

  - Zeros at $z = c_k$ and poles at $z = d_k$ with possible zeros and/or poles at $z = 0$ and $z = \infty$

- Inverse $H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$

➔ The poles (zeros) of $H_i(z)$ are the zeros (poles) of $H(z)$

Pole/zeros relation

- The time-domain condition $h[n] * h_i[n] = \delta[n]$ states ROCs of $H(z)$ and $H_i(z)$ should overlap

- For causal $H(z)$, ROC is $|z| > \max_k |d_k|$

- Any appropriate ROC for $H_i(z)$ that overlaps with $|z| > \max_k |d_k|$ is a valid ROC for $H_i(z)$

Example 1

- Let $H(z) = \dfrac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$ with ROC $|z| > 0.9$
- The inverse becomes $H_i(z) = \dfrac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$
  - Two possible ROCs
  - Only $|z| > 0.5$ overlaps with $|z| > 0.9$
- Impulse response with proper ROC becomes

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

  - The inverse system both causal and stable

Example 2

- Let $H(z) = \dfrac{z^{-1} - 0.5}{1 - 0.9z^{-1}}$ with ROC $|z| > 0.9$
- The inverse becomes $H_i(z) = \dfrac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \dfrac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$
  - Two possible ROCs
  - Both regions overlap with $|z| > 0.9$
- Possible impulse responses

$$h_{i1}[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n] \text{ with ROC } |z| < 2$$

➡ Stable, noncausal

$$h_{i2}[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1] \text{ with ROC } |z| > 2$$

➡ Unstable, causal

Generalization of inverse system

- If causal $H(z)$ has zeros at $c_k, k = 1, \ldots, M$, its inverse will be causal iff the ROC is

$$|z| > \max_k |c_k|$$

- If the inverse $H_i(z)$ to be stable, the ROC of $H_i(z)$ must include unit circle

$$\max_k |c_k| < 1$$

➔ All the zeros of $H(z)$ must be inside unit circle

- If both poles and zeros of $H(z)$ are inside unit circle

➔ Both $H(z)$ and its inverse $H_i(z)$ are causal and stable

➔ Referred to as minimum-phase systems (will be discussed shortly)

FIR vs. IIR Filters

Impulse response for rational system function

- Consider the system with only 1st-order poles (assuming *M*>*N*)

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Assuming the system to be causal

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

IIR vs. FIR systems

- $H(z)$ may have (multiple) poles only at z=0 due to pole/zero cancellations

- If there is at least one nonzero pole of $H(z)$ not cancelled by a zero
  ➔ The impulse response $h[n]$ will have at least one $A_k(d_k)^n u[n]$
  ➔ IIR system

- If $H(z)$ has no poles except at z=0

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}, \;\; h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

  ➔ FIR system

Simple FIR example

- Consider FIR system

$$h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

z=a zero cancels pole

- System function

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}$$

- Input-output relation

$$y[n] = \sum_{k=0}^{M} a^k x[n-k] \qquad y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$

➔ Two expressions represent identical systems

Frequency Response Analysis

Frequency response for rational system functions

- $$H(z) = \frac{z - re^{j\theta}}{z}, \ r < 1$$
- Frequency response

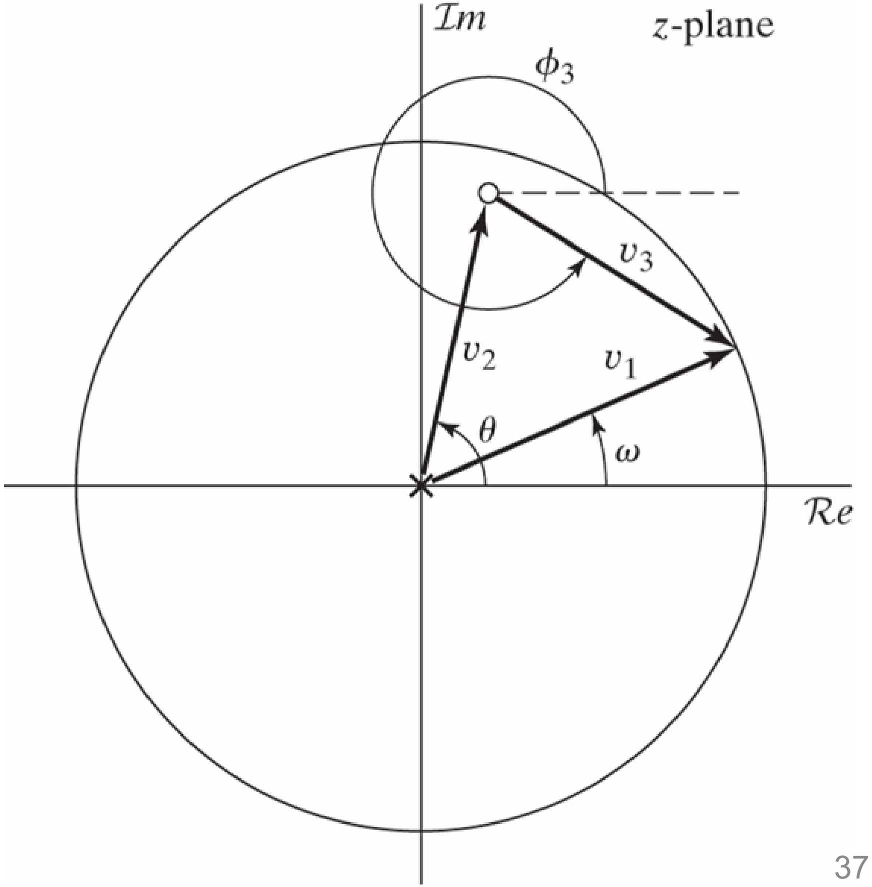
$$H(e^{j\omega}) = \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}$$

- Gain

$$|H(e^{j\omega})| = \frac{|v_3|}{|v_1|}$$

- Phase response

$$\angle H(e^{j\omega}) = \angle(v_3) - \angle(v_1) = \phi_3 - \omega$$



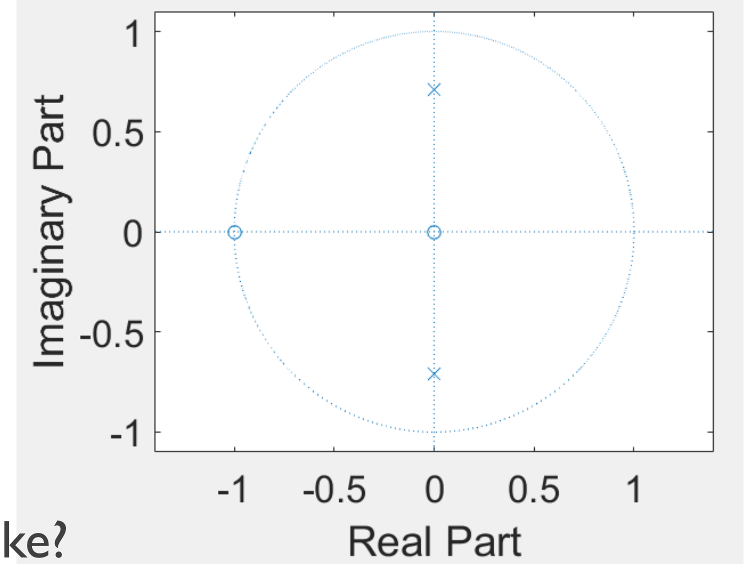
Using poles and zeros to design filters

- Through judicious positioning of zeros and poles
  - Emphasize "desired" frequency bands
  - De-emphasize other frequency bands

- Example:

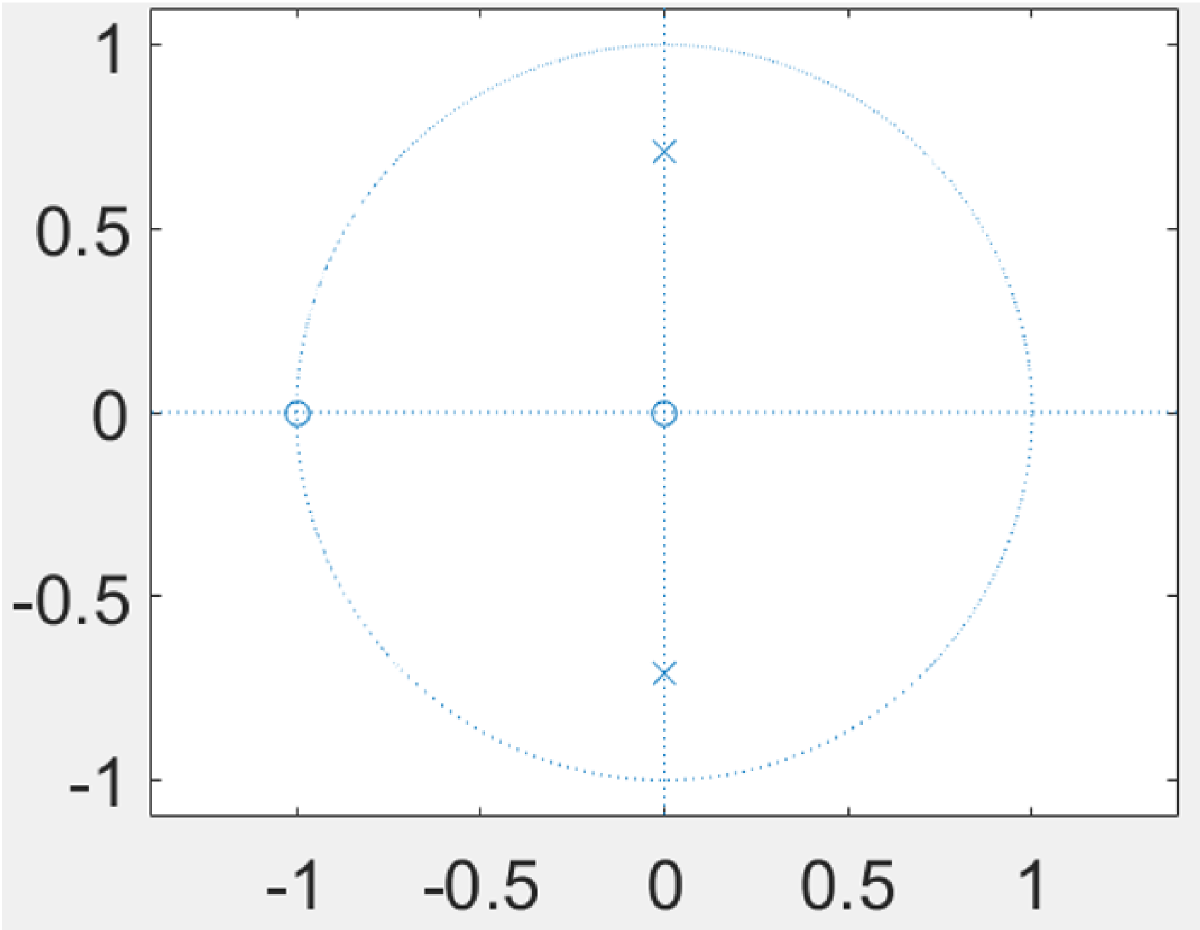
$$y[n] = -\frac{1}{2}y[n-2] + x[n] + x[n-1]$$

$$H(z) = \frac{1+z^{-1}}{1+\frac{1}{2}z^{-2}} = \frac{z(z+1)}{\left(z-j\frac{1}{\sqrt{2}}\right)\left(z+j\frac{1}{\sqrt{2}}\right)}$$



How the frequency response $H(e^{j\omega})$ does look like?

Graphical evaluation of magnitude



$$|H(z)| = \frac{|z||(z+1)|}{\left|z - j\frac{1}{\sqrt{2}}\right|\left|z + j\frac{1}{\sqrt{2}}\right|}$$

All-Pass Filter

All-pass filters

- Mathematical preliminary
  - Let $c = a + jb = |c|e^{j\angle c}$
  - Note $\dfrac{c}{c^*} = \dfrac{|c|e^{j\angle c}}{|c|e^{-j\angle c}} = 1e^{j2\angle c}$ ➔ $\left|\dfrac{c}{c^*}\right| = 1$
- Consider the system with single pole at $z = p$ and a single zero at $z = \dfrac{1}{p^*}$

$$H(z) = G\frac{z - \frac{1}{p^*}}{z - p}$$

- Frequency response becomes

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = G\frac{e^{j\omega} - \frac{1}{p^*}}{e^{j\omega} - p} = -\frac{G}{e^{j\omega}p}\frac{c}{c^*}, \text{ where } c = e^{j\omega} - \frac{1}{p^*}$$

All-pass filters

- The amplitude of frequency response

$$|H(e^{j\omega})| = \left|\frac{G}{p}\right| = \frac{|G|}{|p|}$$

does not depend on ω ➔ all-pass filter!

- An all-pass filter can be used to stabilize an unstable system without affecting the magnitude of the frequency response
  - Example: assume $|p| < 1$

Everything except $z - 1/p^*$

$$\boxed{\frac{H'(z)}{z - \frac{1}{p^*}}} \times \frac{z - \frac{1}{p^*}}{z - p}$$

Original system

The original pole $z = 1/p^*$ outside unit circle

The new pole z=p inside unit circle

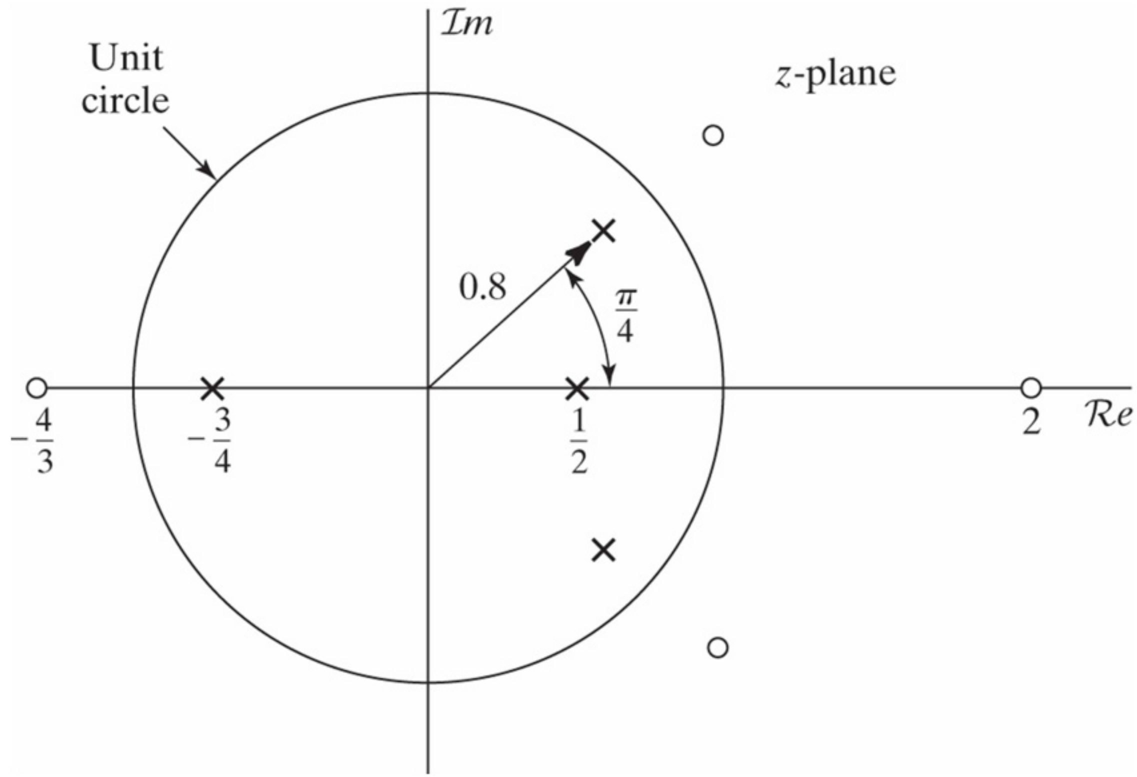
➔ The overall system becomes stable!

➔ Magnitude is unaffected

Typical pole-zero plot of all-pass system

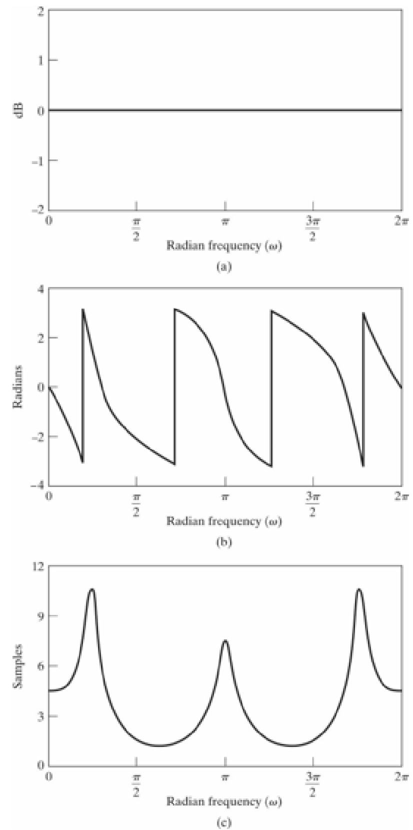
- All poles inside unit circle
- Each pole has a conjugate reciprocal zero

$$H(z) = G\frac{z - \frac{1}{p^*}}{z - p}$$



Frequency response of all-pass system

Figure 5.21 Frequency response for an all-pass system with the pole–zero plot in Figure 5.18. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.



Minimum-Phases Systems

Minimum-phases systems

- To have causal and stable systems
  ➔ Poles must be inside the unit circle but no restriction on zeros

- To have causal and stable inverse
  ➔ Zeros must be inside the unit circle as well

- If all poles and zeros are inside the unit circle
  ➔ Such systems are referred to as minimum-phases systems

Minimum-phase and all-pass decomposition

- Any causal rational function can be decomposed as

$$H(z) = \underbrace{H_{\min}(z)}_{\text{Minimum-phase system}} \underbrace{H_{\mathrm{ap}(z)}}_{\text{All-pass system}}$$

- Proof: suppose $H(z)$ has one zero outside the unit circle at $z = 1/c^*$ and remaining poles and zeros are inside the unit circle

$|c| < 1$

$$H(z) = \underbrace{H_1(z)}_{\text{Minimum-phase system}}(z^{-1} - c^*) = \underbrace{H_1(z)(1 - cz^{-1})}_{\text{Minimum-phase system}} \underbrace{\frac{z^{-1} - c^*}{(1 - cz^{-1})}}_{\text{All-pass system}}$$

- Possible to generalize to multiple zeros outside the unit circle

Important property

- Let $H(z) = H_{\min}(z) H_{\mathrm{ap}(z)}$

- Frequency-response relationship

$$|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|$$

for all ω because

$$|H_{\mathrm{ap}}(e^{j\omega})| = 1$$

for all ω

Example 1

- Decompose $H_1(z) = \dfrac{(1+3z^{-1})}{1+\frac{1}{2}z^{-1}}$ → Zero at z=3 ➔ outside unit circle
- Use reciprocal zero, i.e., z=1/3, for minimum-phase system

$$H_{\min}(z) = 3\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

- Design appropriate all-pass system

$$H_{\mathrm{ap}}(z) = \frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}$$

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Example 2

Two zeros outside unit circle

- Decompose $H_2(z) = \dfrac{\left(1+\frac{3}{2}e^{+j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$

$$= \frac{9}{4}\frac{\left(z^{-1}+\frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1}+\frac{2}{3}e^{j\pi/4}\right)}{1-\frac{1}{3}z^{-1}}$$

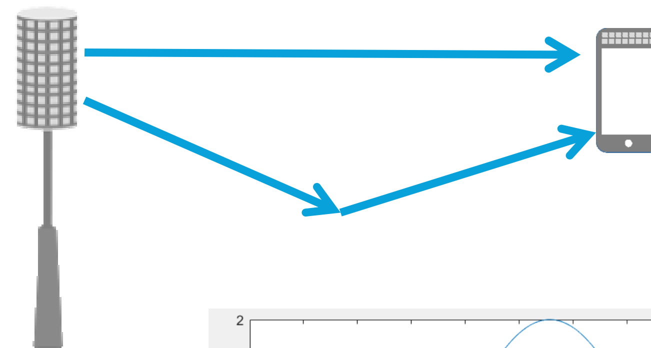
$$H_2(z) = \left[\frac{9}{4}\frac{\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\right]$$

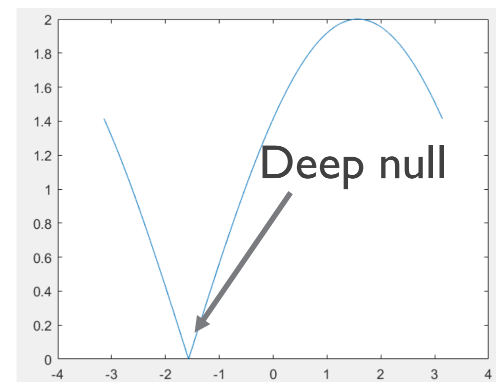
$$\times\left[\frac{\left(z^{-1}+\frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1}+\frac{2}{3}e^{j\pi/4}\right)}{\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)}\right]$$

Frequency-response compensation

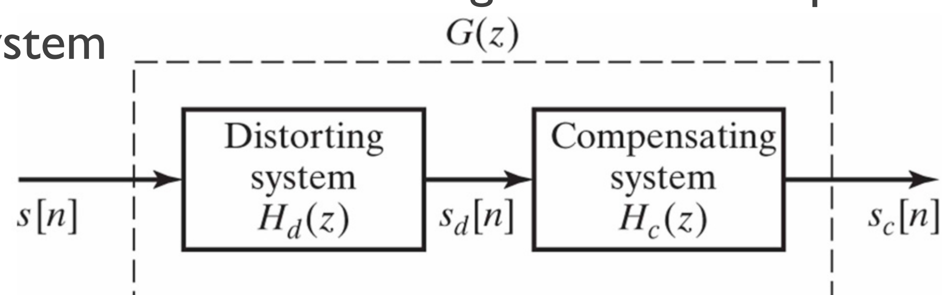
- Signals can be distorted by an LTI system with an undesirable frequency response
  - Example: two-path communication channel

$$q(t) = \underbrace{\delta(t) - e^{j\phi}\delta(t - T_0)}_{\text{Two-way multipath}}$$

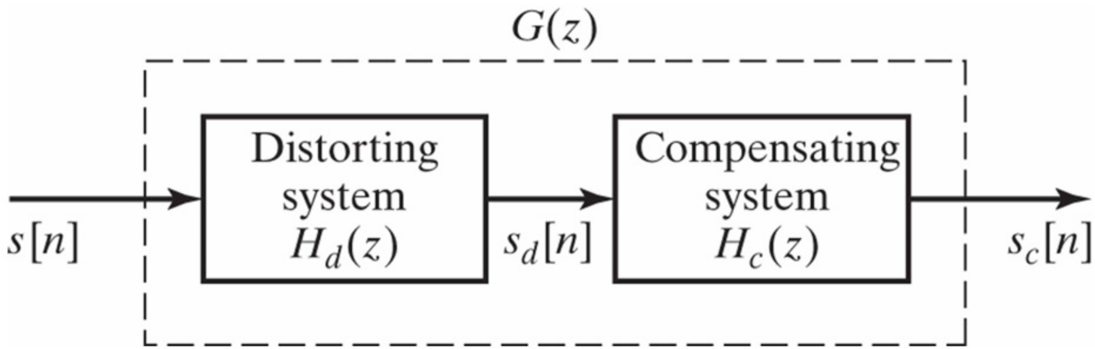




- Process the distorted signal with a compensating system



Perfect compensation



- With perfect compensation $s_c[n] = s[n]$
  ➔ $H_c(z)$ is the inverse of $H_d(z)$

- We are interested in stable and causal distorting and compensating systems
  ➔ Perfect compensation is possible only if $H_d(z)$ is a minimum-phase system

- Not all distorting systems are minimum-phase systems

Non-ideal compensation systems

- Decompose $H_d(z) = H_{d\,\min}(z) H_{\text{ap}}(z)$

  Minimum-phase system　　All-pass system

- Choose the compensating filter

$$H_c(z) = \frac{1}{H_{d\,\min}(z)}$$

- Overall system becomes

$$G(z) = H_d(z) H_c(z) = H_{\text{ap}}(z)$$

  - Frequency-response magnitude exactly compensated
  - Phase response modified to $\angle H_{\text{ap}}(e^{j\omega})$

FIR system example

- Consider $H_d(z) = (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})$
  $\times (1 - 1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$

  Zeros outside unit circle

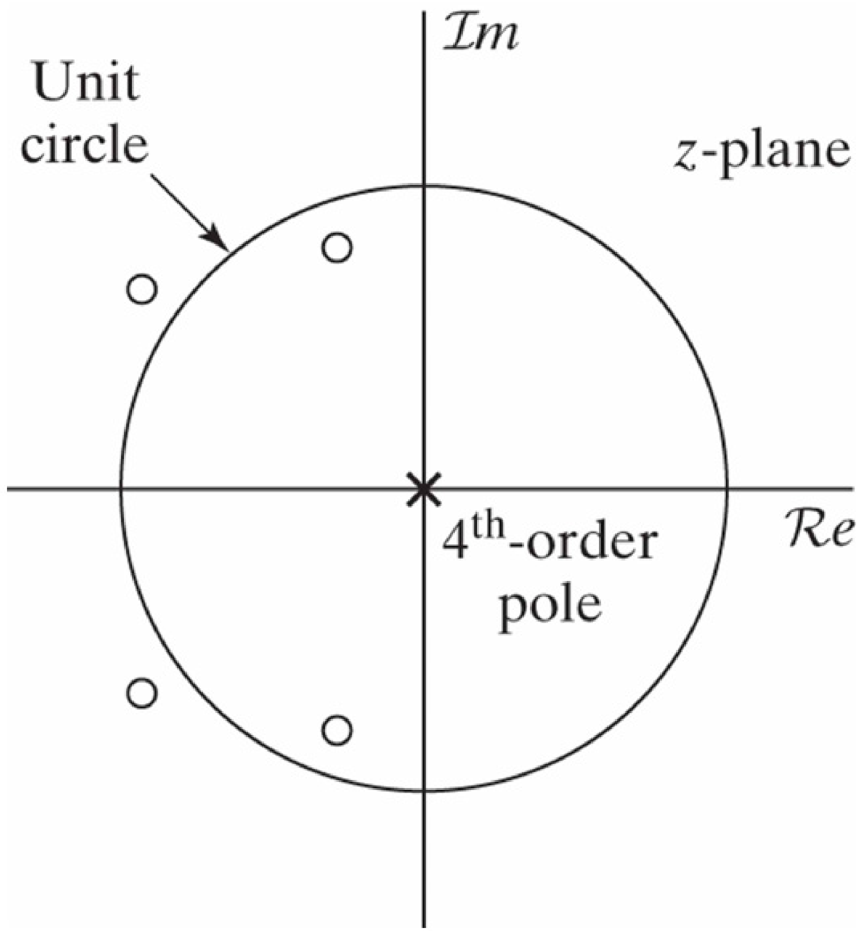
- Rewrite as $H_d(z) = (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})(1.25)^2$
  $\times (z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})$
- Minimum-phase system

$$H_{\text{min}}(z) = (1.25)^2(1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \times (1 - 0.8e^{-j0.8\pi}z^{-1})(1 - 0.8e^{j0.8\pi}z^{-1})$$

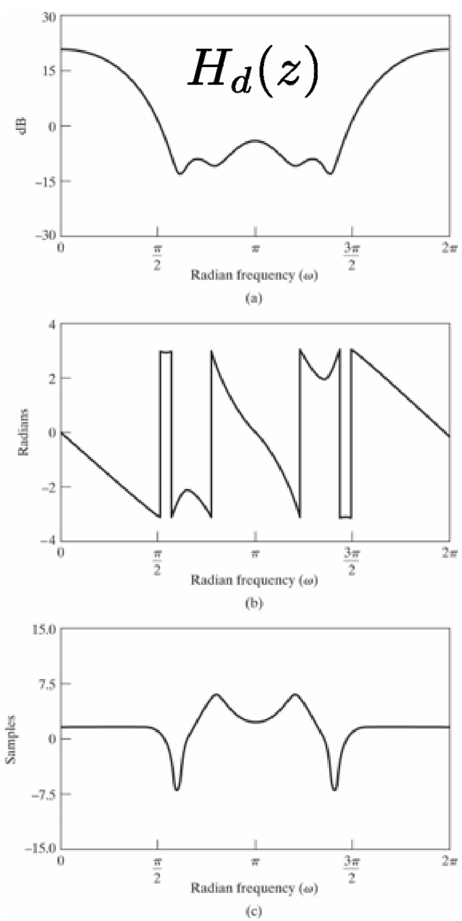
- All-pass system

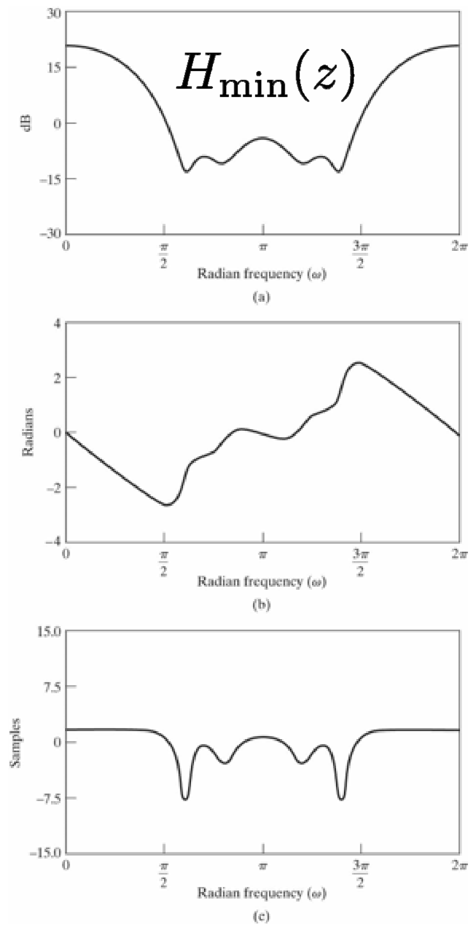
$$H_{\text{ap}}(z) = \frac{(z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})}{(1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1})}$$

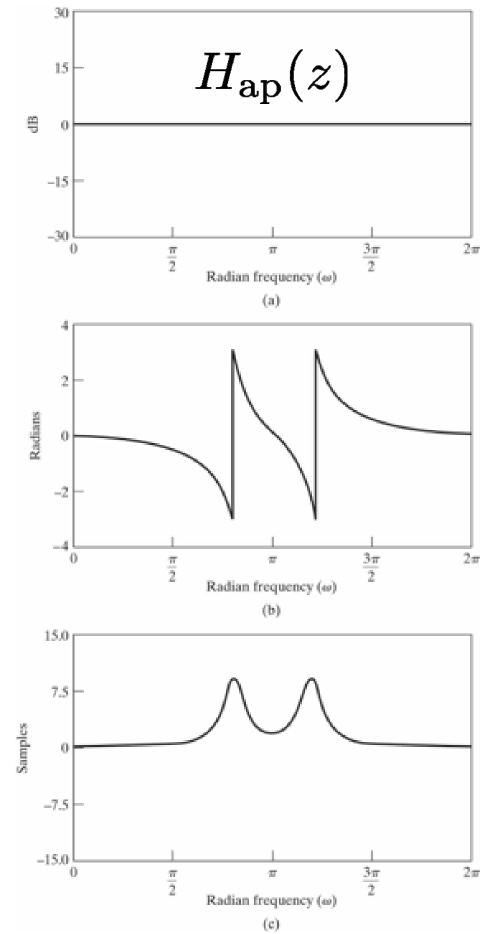
Pole-zero plot



Frequency-response plots

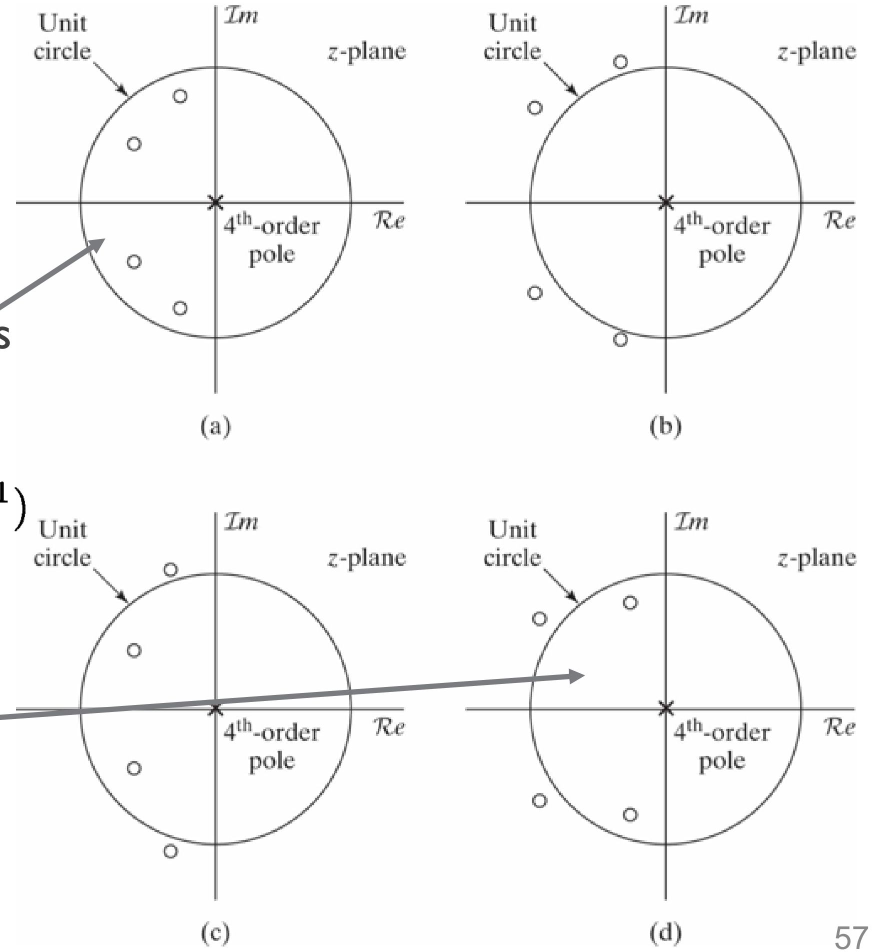






Properties of minimum-phase system I

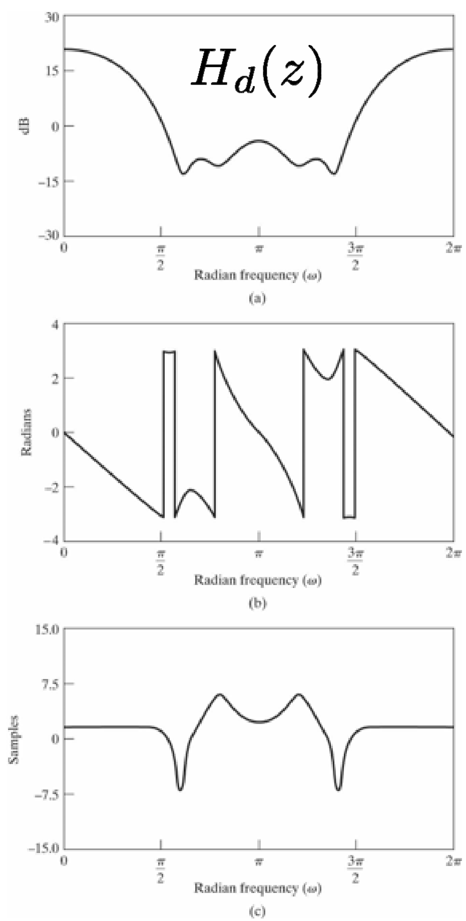
- For a system with M pairs of zeros
  - 2^M possible causal & stable systems with the same frequency-response magnitude $|H(e^{j\omega})|$
  - Only one minimum-phase system exists
    ➔ All zeros inside unit circle

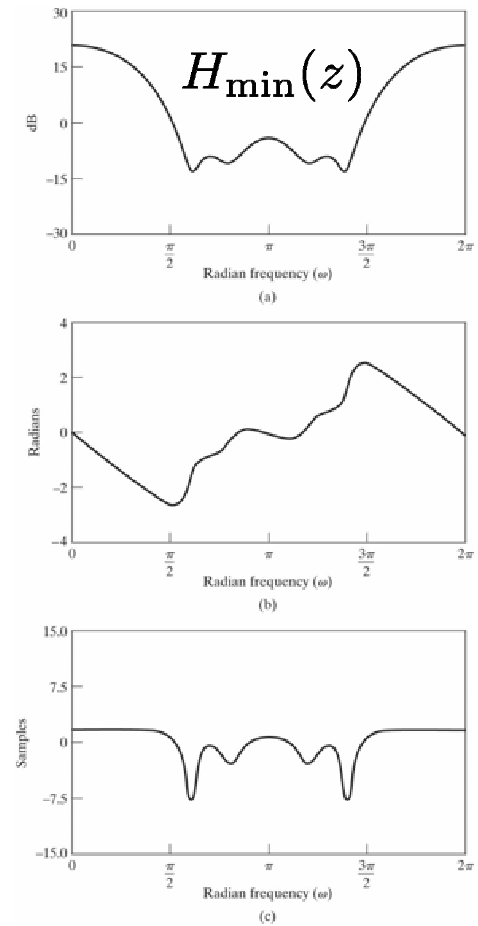


$$H_{\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1}) \times (1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

$$H_d(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1}) \times (1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$$

Frequency-response plots





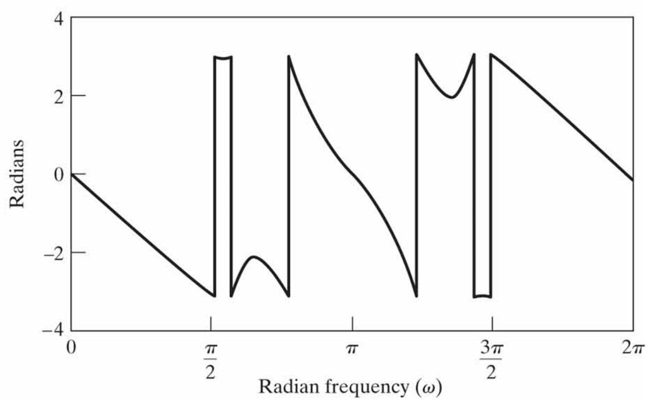
Minimum phase-lag property

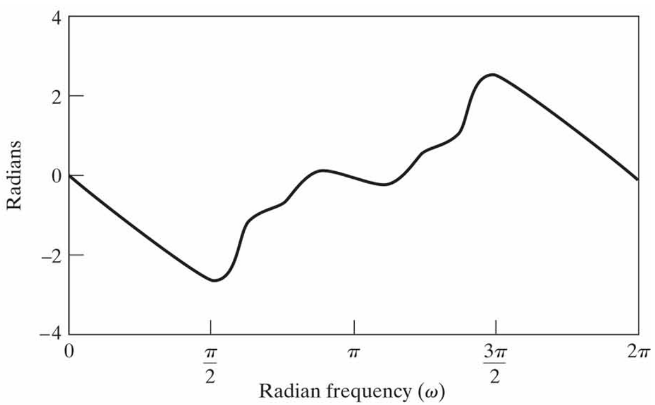
- Define the negative of the phase as "phase-lag"
  ➔ Larger the phase, smaller the phase-lag

Always negative in $0 \leq \omega \leq \pi$

- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the minimum phase-lag function

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{\text{ap}}(e^{j\omega})]$$



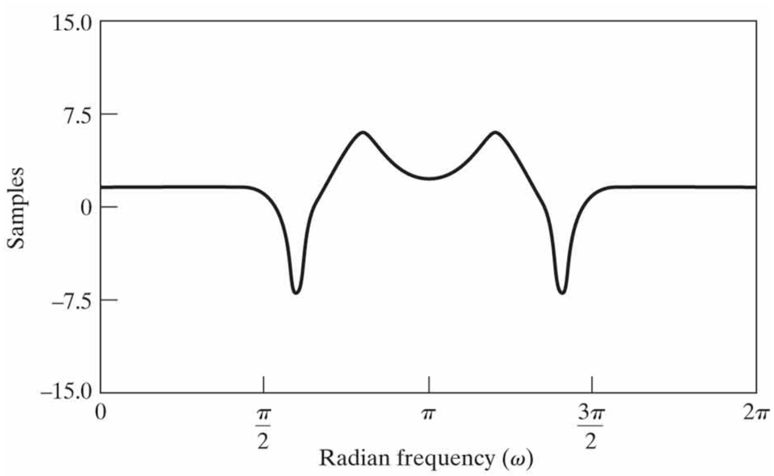


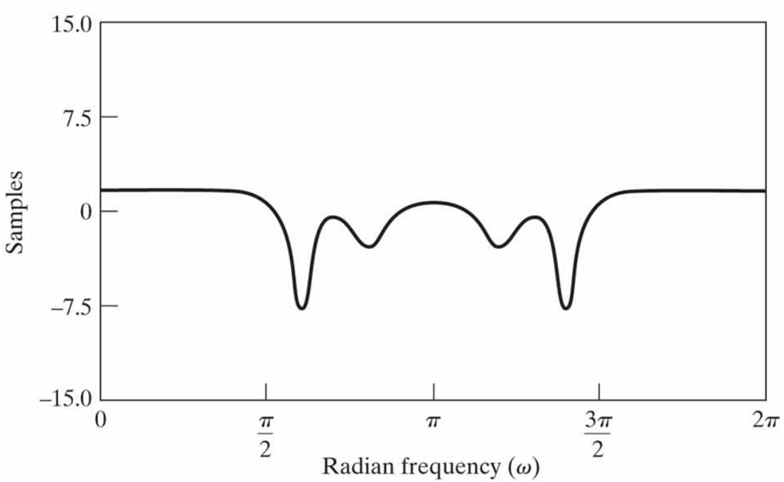
Minimum group-delay property

- Clearly $\mathrm{grd}[H(e^{j\omega})] = \mathrm{grd}[H_{\mathrm{min}}(e^{j\omega})] + \mathrm{grd}[H_{\mathrm{ap}}(e^{j\omega})]$

Always positive in $0 \leq \omega \leq \pi$

- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the minimum group delay





Minimum energy-delay property

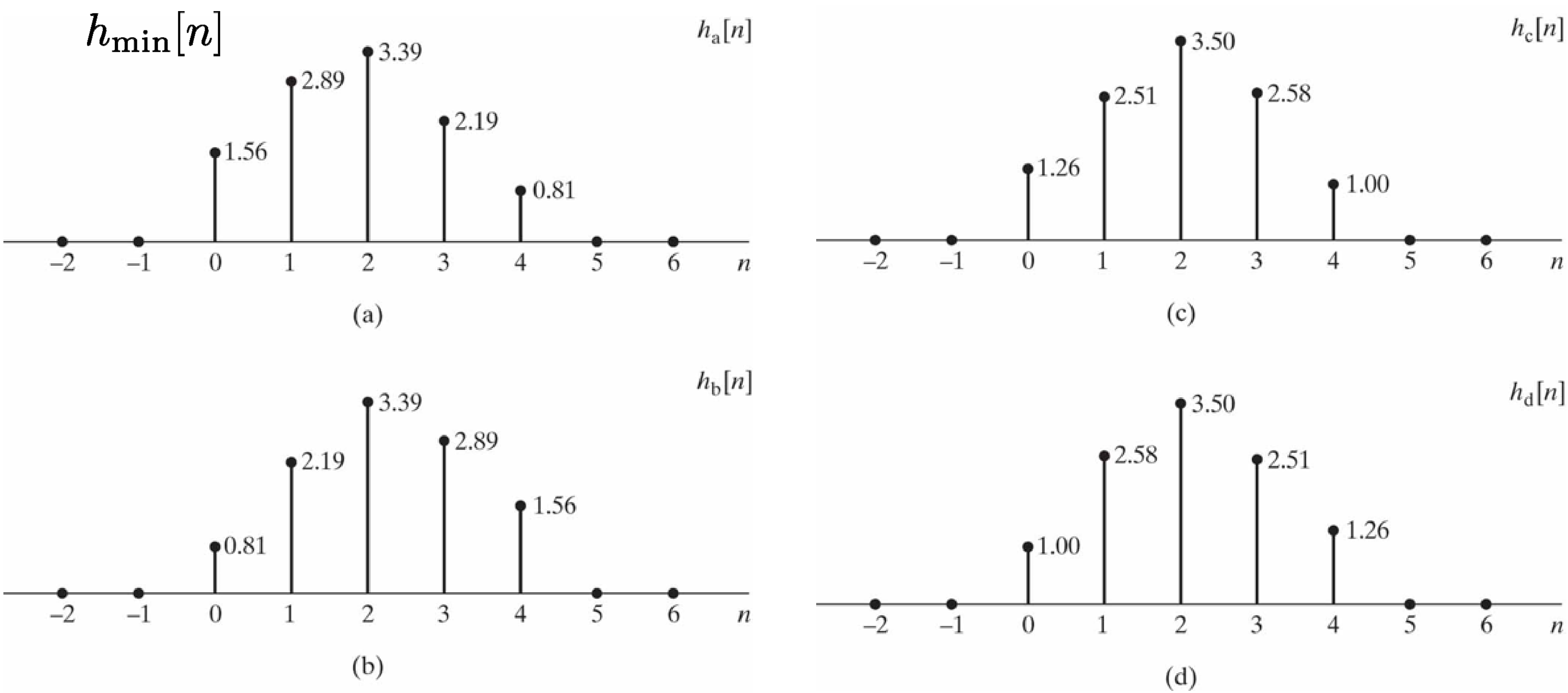
- All systems that have the same frequency-response magnitude has equal energy

$$\sum_{n=0}^{\infty}|h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}|H_{\min}(e^{j\omega})|^2 d\omega = \sum_{n=0}^{\infty}|h_{\min}[n]|^2$$

- Define partial energy $E[n] = \sum_{m=0}^{n}|h[m]|^2$
- Of all systems with the same $|H(e^{j\omega})|$, the system with all poles and zeros inside the unit circle has the most energy concentrated around n=0

$$\sum_{m=0}^{n}|h[m]|^2 \leq \sum_{m=0}^{n}|h_{\min}[m]|^2$$

Minimum energy-delay property



Linear phase systems

- For causal systems, zero phase is not possible
  - Some phase distortion must be allowed
- In many situations, it is desirable to design systems to have exactly or approximately linear phase
- Ideal delay system example

$$H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega\alpha}, \ |\omega| < \pi$$

$$|H_{\mathrm{id}}(e^{j\omega})| = 1$$

$$\angle H_{\mathrm{id}}(e^{j\omega}) = -\omega\alpha$$

$$\mathrm{grd}[H_{\mathrm{id}}(e^{j\omega})] = \alpha$$

  - α does not have to be an integer (See 5.7.1)

Generalized linear phase

- Generalized linear-phase system is defined as

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$

$$\alpha, \beta : \text{ real constants}$$

$$A(e^{j\omega}) : \text{ a real function of } \omega$$

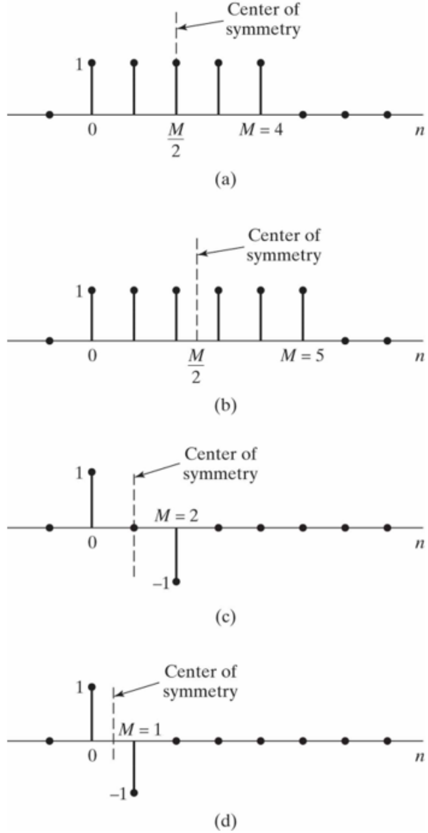
- Phase and group delay

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha$$

$$\text{grd}[H(e^{j\omega})] = \alpha$$

Causal FIR generalized linear-phase systems

- Four classes of FIR systems with generalized linear-phase
  - Type I
    - Symmetric: $h[n] = h[M-n]$
    - M even
  - Type II
    - Symmetric: $h[n] = h[M-n]$
    - M odd
  - Type III
    - Antisymmetric: $h[n] = -h[M-n]$
    - M even
  - Type IV
    - Antisymmetric: $h[n] = -h[M-n]$
    - M odd



Locations of zeros for FIR linear-phase systems

- For Types I and II, channel impulse responses are symmetric $h[n] = h[M-n]$
- System function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M} = z^{-M}H(z^{-1})$$

- If z_0 is a zero of $H(z)$, then $H(z_0) = z_0^{-M}H(z_0^{-1}) = 0$
  ➔ z_0^{-1} is also a zero
- If h[n] is real and z_0 is a zero of $H(z)$, then z_0^* is also a zero
  ➔ $(z_0^*)^{-1}$ is also a zero

- $H(z)$ will have factors of the form $(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1})$
  - What if zeros are on the unit circle?

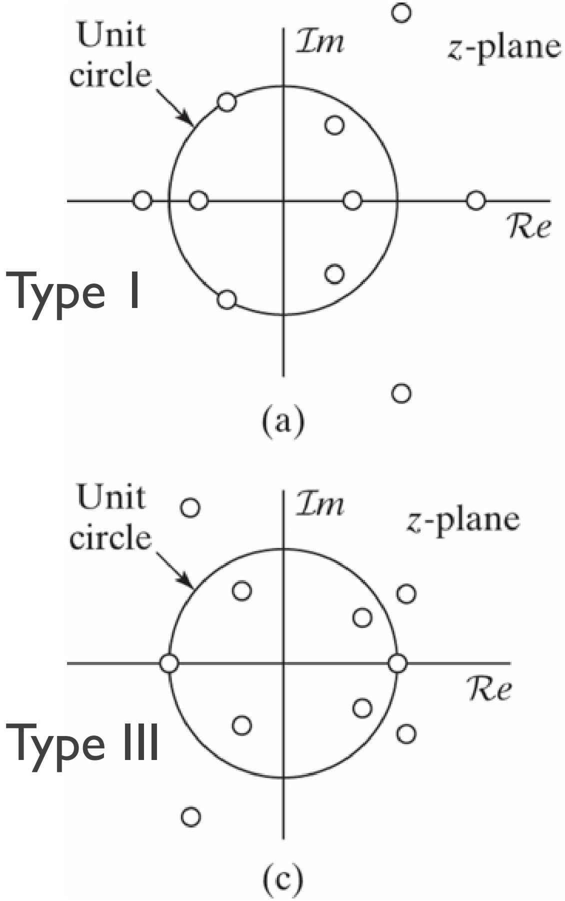
Locations of zeros for FIR linear-phase systems

- For Types III and IV, $h[n] = -h[M-n]$
- System function $H(z) = -z^{-M}H(z^{-1})$

- If z=1, $H(1) = -H(1)$ ➔ z=1 is always a zero
- If z=-1, $H(-1) = (-1)^{-M+1}H(-1)$
  ➔ If M is even, z=-1 should be a zero

- These constraints are important in FIR linear-phase filter designs
  - Example: with symmetric impulse response, $z = -1 \ (\omega = \pi)$ should be always zero with M odd
    ➔ For highpass filter with symmetric impulse response, M should be even!

Typical plots of zeros for linear-phase systems



IIR filter and linear-phase

- So far, we discussed FIR linear-phase filters
- Can IIR filters have a linear-phase response?

- Check with the same criterion

$$H(z) = \pm z^{-M} H(z^{-1})$$

  - If p_0 is a pole of $H(z)$, then $1/p_0$ is also a pole
  - If h[n] is real, then p_0^* and $1/p_0^*$ are also poles

  ➔ Cannot be causal and stable!!!