Digital Signal Processing

POSTECH

Department of Electrical Engineering

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POSTECH

Auto/cross-correlations of Deterministic Signals

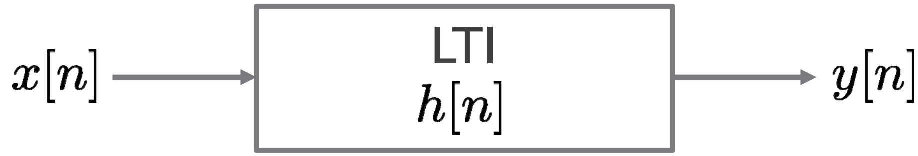
Properties of autocorrelation sequences

- Three main properties of autocorrelation sequence

$$c_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

- $c_{xx}[-\ell] = c_{xx}^*[\ell]$
- $|c_{xx}[\ell]| \leq c_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \text{energy}$
- $\sum_{\ell=-\infty}^{\infty} c_{xx}[\ell] e^{-j\omega\ell} \geq 0, \text{ for all } \omega$

Input/output relationships for LTI system



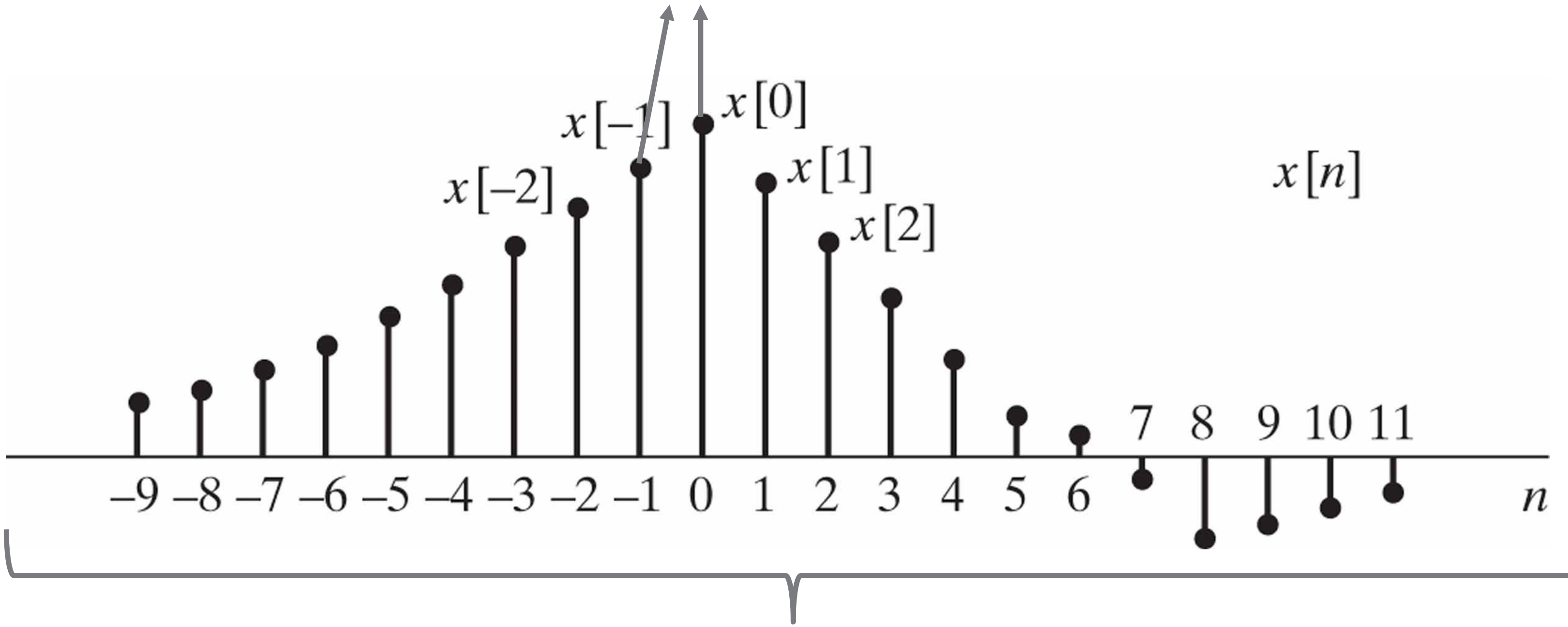
- $c_{yx}[\ell] = c_{xx}[\ell] * h[\ell]$
- $c_{yy}[\ell] = c_{xx}[\ell] * c_{hh}[\ell]$

Try to prove these

Discrete-Time Random Signals

Discrete-time random signals

Each sample is a random variable
following a certain probability density function



Collection of random variables is referred to as a random process

Wide-sense stationary (WSS) random signals

- The mean of random process x[n] at time n is

$$m_x[n] = \mathcal{E}\{x[n]\}$$

- The autocorrelation of x[n] is

$$\phi_{xx}[n, n+m] = \mathcal{E}\left\{x[n]x[n+m]\right\}$$

- If x[n] is wide-sense stationary random process, then

$$m_x[n] = m_x, \quad \text{for all } n$$

$$\phi_{xx}[n, n+m] = \phi_{xx}[m], \quad \text{for all } n \text{ and } m$$

Input-output relation of LTI systems

- Recall a LTI system $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
- With WSS input, the mean of output becomes

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\} = \boxed{m_x \sum_{k=-\infty}^{\infty} h[k]}$$

Constant!

  - Alternative expression

$$m_y = H(e^{j0})m_x$$

Input-output relation of LTI systems

- Autocorrelation of output is

$$
\begin{aligned}
\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\} \\
&= \mathcal{E}\left\{ \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r] \right\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\{x[n-k]x[n+m-r]\} \\
&= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] \\
&= \phi_{yy}[m]
\end{aligned}
$$

WSS input

Not a function of *n*!

Fourier transform of autocorrelations

- Let $\phi_{xx}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Phi_{xx}(e^{j\omega}),\ \phi_{yy}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Phi_{yy}(e^{j\omega}),\ c_{hh}[n] \overset{\mathcal{F}}{\longleftrightarrow} C_{hh}(e^{j\omega})$
- Because $\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell]$

$$c_{hh}[\ell] = h[n] * h[-n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2$$

we have
$$\begin{aligned}\Phi_{yy}(e^{j\omega}) &= C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}) \\ &= |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega})\end{aligned}$$

Power density spectrum or power spectrum density

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})d\omega$$

MATLAB Programming

DTFT computation - preliminary

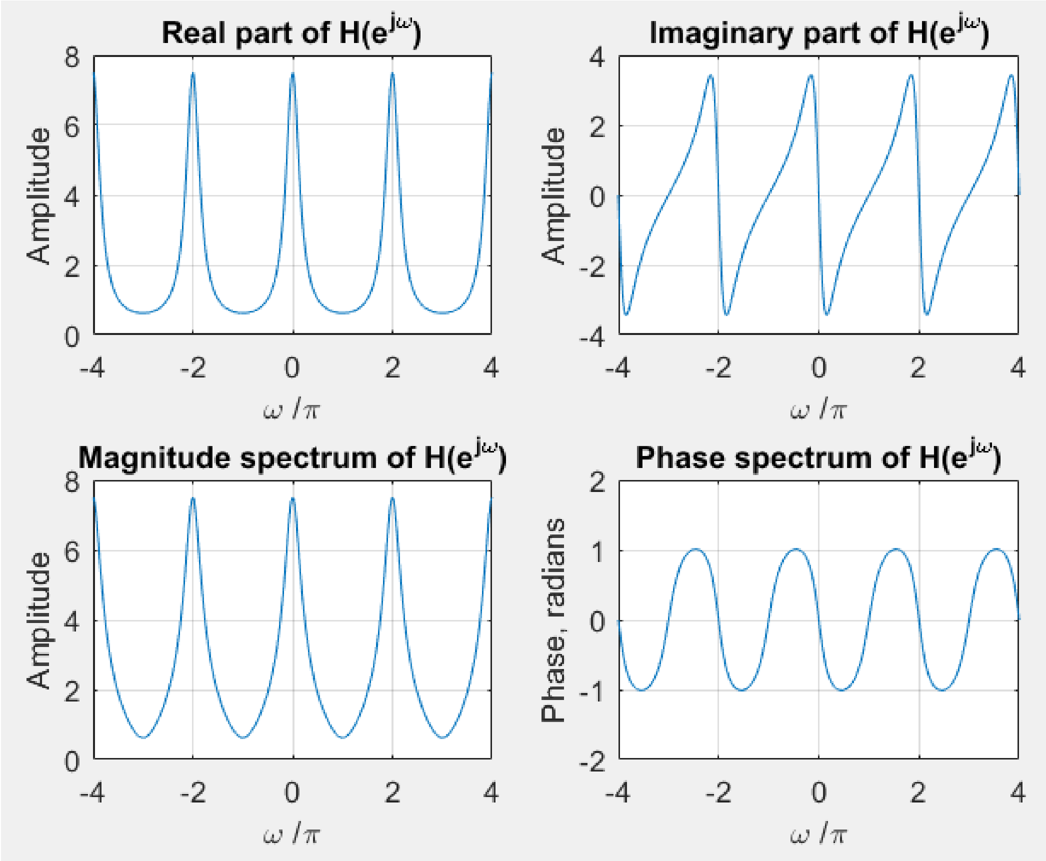
- Use 'freqz' function (different from fft function)
- Only can evaluate DTFT that is expressed as

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$$

DTFT computation

```
num=[2 1];
den=[1 -0.6];
w=-4*pi:8*pi/511:4*pi;
h=freqz(num,den,w);

subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude spectrum of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Amplitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase spectrum of H(e^{j\omega})')
xlabel('\omega /\pi')
ylabel('Phase, radians')
```

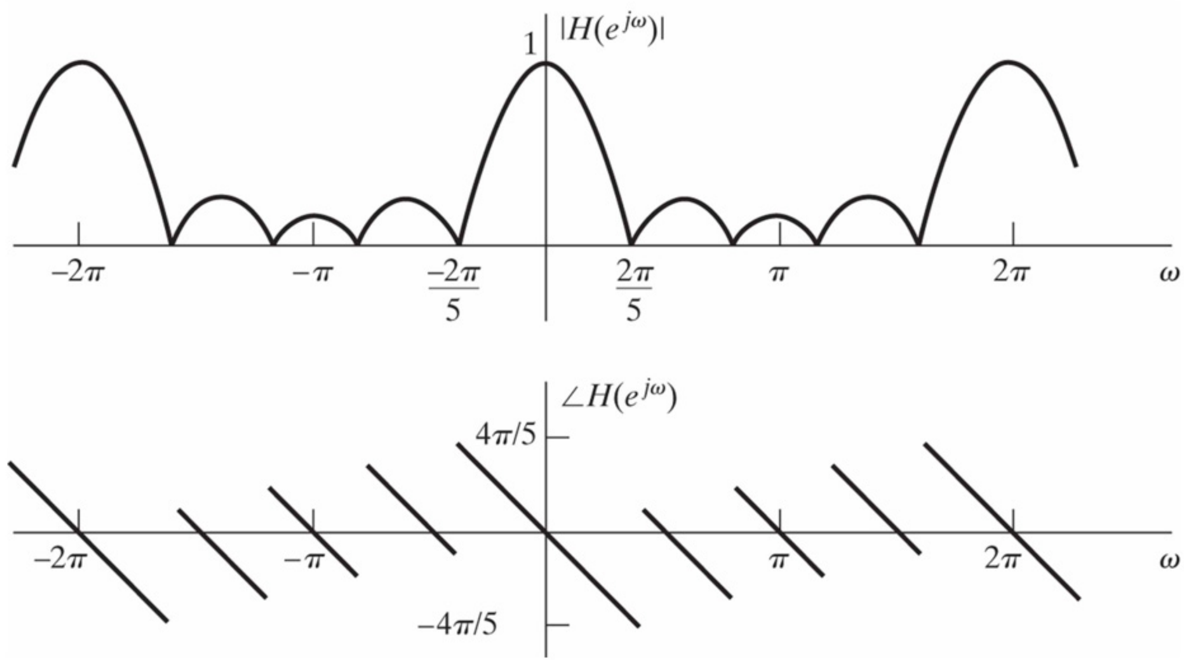


Announcements

- Homework
  - Problems in textbook: 2.36, 2.37, 2.43, 2.45, 2.58
  - MATLAB problems
    - Submit by 10/4 (Thursday) before the class
      - Make a document containing relevant figures and answers for questions
        - No hard copy required.
      - Send a zip file containing all m-files, plots, and the document.
      - Please refer to the instructions in the first MATLAB homework
        - Annotate only some key lines from now on
- First quiz on 10/2 (Tuesday)
  - Will have this quiz in the beginning of class (so, don't be late!)
  - Coverage will be the first and second textbook homework problems
  - Around 3~4 questions for 20 minutes
  - Will talk about your possible future careers after the quiz in Korean

MATLAB problem 1

- Plot the graphs in Example 2.16 using equation (2.123) with $M_2 = 4$
- Use 'freqz' to plot the same graph



MATLAB problem 2

- Time-shift property of DTFT is

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) \Rightarrow x[n-n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} X(e^{j\omega})$$

- Using 'freqz' function, write a MATLAB program to check time-shift property
  - $x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$ with $n_d = 5$
  - Plot magnitudes and phases of original and delayed sequences
  - Use 'unwrap' function to remove sudden jumps in phases
  - Interpret the phase plots to show time-shift property holds

MATLAB problem 3

- Convolution property of DTFT is

$$x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

- Using 'freqz' function, write a MATLAB program to check convolution property
  - Use

$$x[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10]), y[n] = (3)^n (u[n] - u[n-7])$$

  - You should compare the Fourier transforms of $x[n] * y[n]$ and $X(e^{j\omega})Y(e^{j\omega})$
  - Plot both magnitudes and phases

MATLAB problem 4 (require 6 plots total)

- Consider the radar example

$$y[n] = \Gamma s[n-D] + w[n] = s[n] * \Gamma\delta[n-D] + w[n]$$

- Define variables

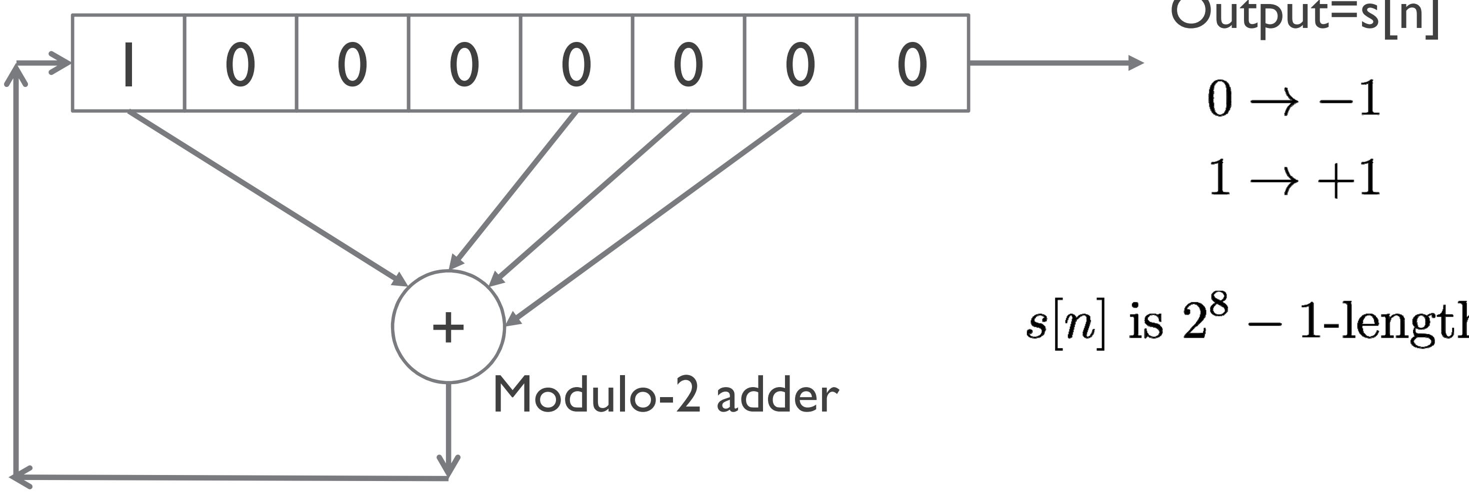
  $s[n] = \{1, 1, 1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1\}, \ \Gamma = 0.9, \ D = 20$

  $w[n]:$ Gaussian random sequence with zero mean and variance $\sigma^2 = 0.01$

- Write a program to plot y[n] for $0 \leq n \leq 199$
- Compute and plot the cross-correlation $c_{ys}[\ell]$ for $0 \leq \ell \leq 59$
  - Can we estimate D with this cross-correlation?
- Repeat all parts with $\sigma^2 = 0.1$ and $\sigma^2 = 1$
  - What is the role of σ^2 in finding D? Is it helpful or not?

MATLAB problem 5 (require 6 plots total)

- Repeat "MATLAB problem 4" with s[n] obtained from the linear shift register



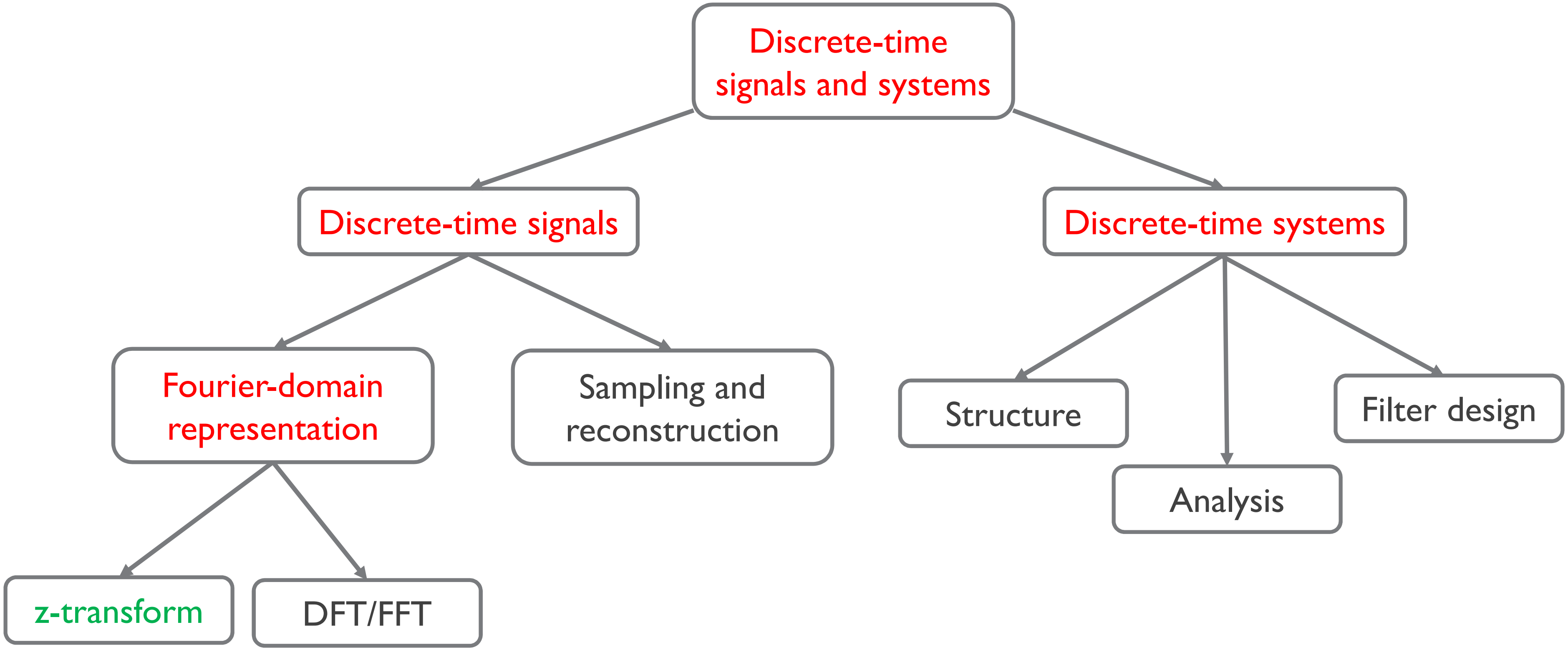
$$0 \rightarrow -1$$

$$1 \rightarrow +1$$

$s[n]$ is $2^8 - 1$-length sequence

- When plotting y[n], set $0 \leq n \leq 500$. For $c_{ys}[\ell]$, set $0 \leq \ell \leq 99$
- Is having a longer sequence s[n] beneficial to detect the target in radar?

Course at glance



The z-Transform

Preliminaries

- Transforms
  - Fourier transform
  - Laplace transform

  } Continuous time

  - Discrete-time Fourier transform (DTFT)
  - z-transform

  } Discrete time

- z-transform is a generalization of DTFT

Limitation of Fourier transform

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Sufficient condition for the existence of DTFT

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- DTFT may not exist for sequences that are not absolutely summable

➡ Require more generalized transform

z-transform definition

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform (can interpret as a function of z)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \Rightarrow \quad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

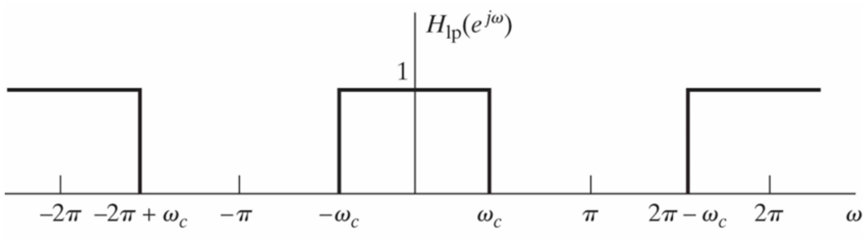
- The complex variable z in polar form $z = re^{j\omega}$

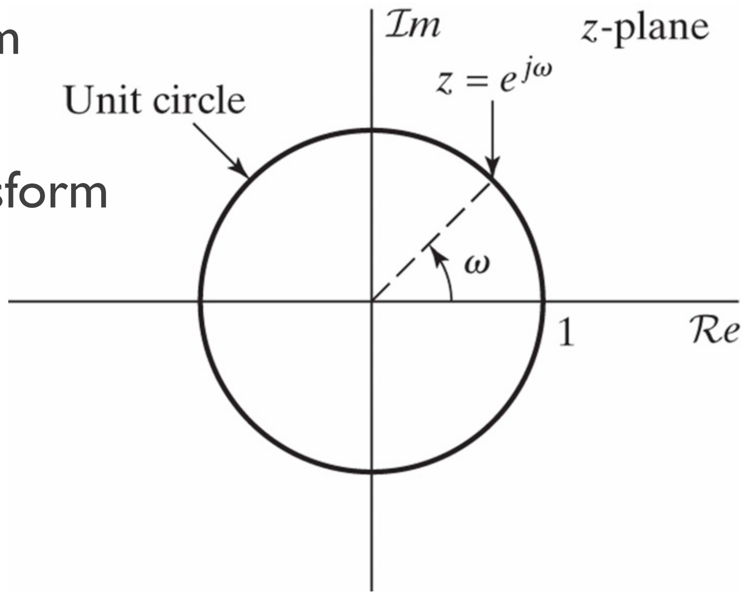
$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- If $|z| = r = 1, \quad X(z) = X(e^{j\omega})$

z-plane

- z-transform is a function of complex variable z
  - ➡ Interpret using the complex z-plan
- z-transform on unit circle = Fourier transform
- Linear frequency axis (x-axis) in Fourier transform
  - ➡ Unit circle in z-transform with period 2π





Region of convergence (ROC)

- DTFT does not converge for all sequences (depends on x[n])

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform does not converge for all sequences OR for all values of z

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- ROC: **for any given sequence x[n],** the set of values of z for which z-transform converges

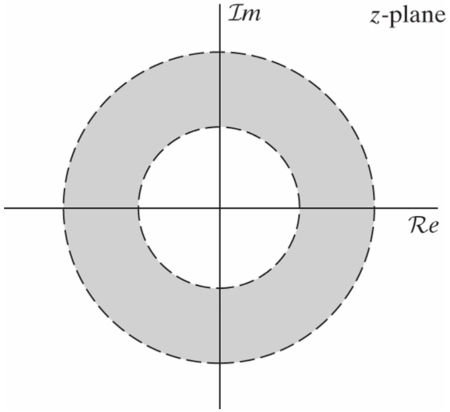
$$|X(re^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| \leq \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

Shape of ROC

- Convergence depends only on $|z|$
  - If a point $z = z_1$ is part of ROC, then the circle with $|z| = |z_1|$ is part of ROC

- ROC consists of a ring centered at the origin
  - Outer boundary is a circle (may extend to infinity)
  - Inner boundary is a circle (may extend inward to include the origin)

- If ROC includes the unit circle $|z| = 1$
  - ➡ DTFT exists!



ROC example

Right-sided sequence

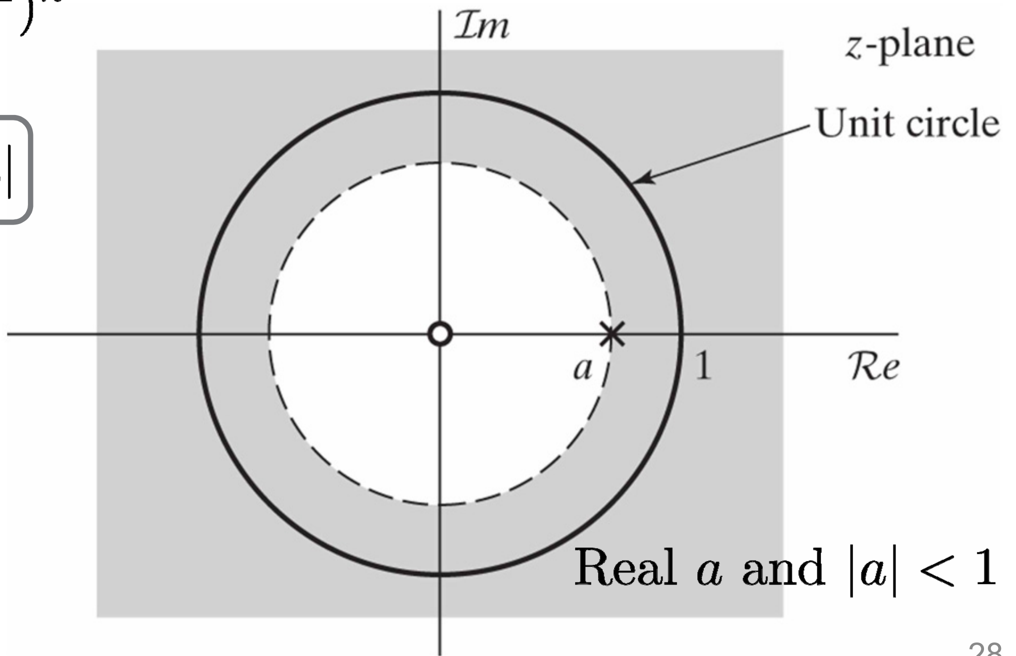
- Consider sequence $x[n] = a^n u[n]$
- z-transform of x[n]

$|az^{-1}| < 1$ to converge

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad \boxed{|z| > |a|}$$

ROC



Real a and $|a| < 1$

Zeros and poles

- Most important and useful z-transform expression – rational function of z

$$X(z) = \frac{P(z)}{Q(z)}$$

  - In previous example,

$$X(z) = \frac{z}{z-a}$$

- Zeros of $X(z)$: the values of z for $X(z) = 0$
  - ➡ Roots of polynomial $P(z)$

- Poles of $X(z)$: the values of z for $X(z) = \infty$
  - ➡ Roots of polynomial $Q(z)$

ROC example 1

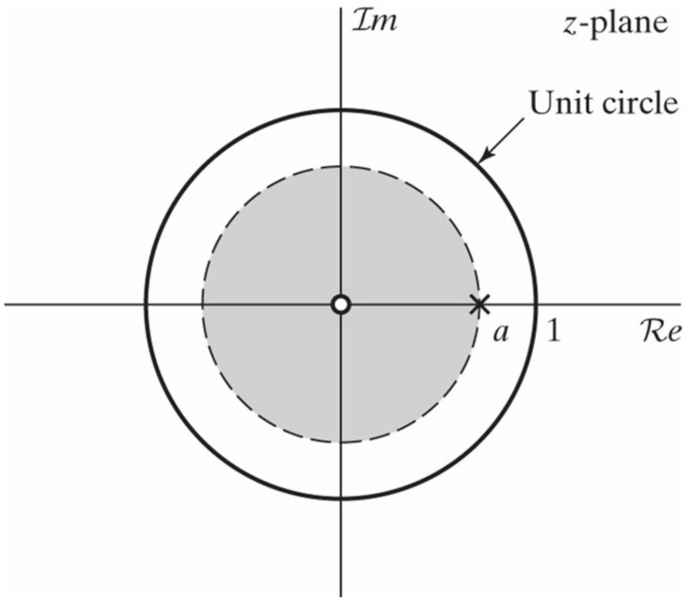
- Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \leq -1 \\ 0 & n > -1 \end{cases}$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n = 1 - \frac{1}{1 - a^{-1} z} = \frac{z}{z-a}$$

with ROC $|z| < |a|$



ROC example 2

- Sum of two exponential sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

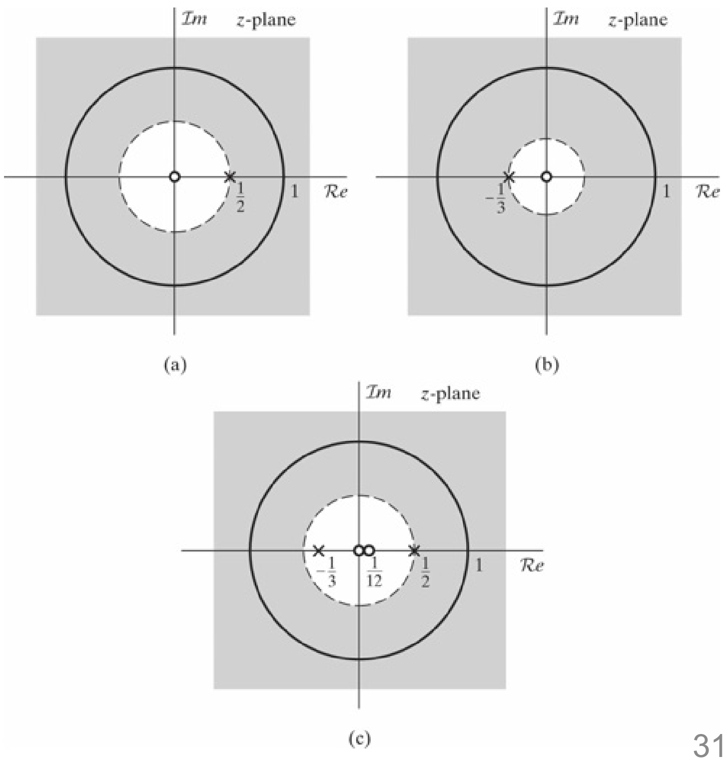
$$\text{ROC:}|z| > \frac{1}{2} \qquad \text{ROC:}|z| > \frac{1}{3}$$

Use linearity of z-transform

$$X(z) = \cdots = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

$$\text{ROC:}|z| > \frac{1}{2} \text{ and } |z| > \frac{1}{3}$$



ROC example 3

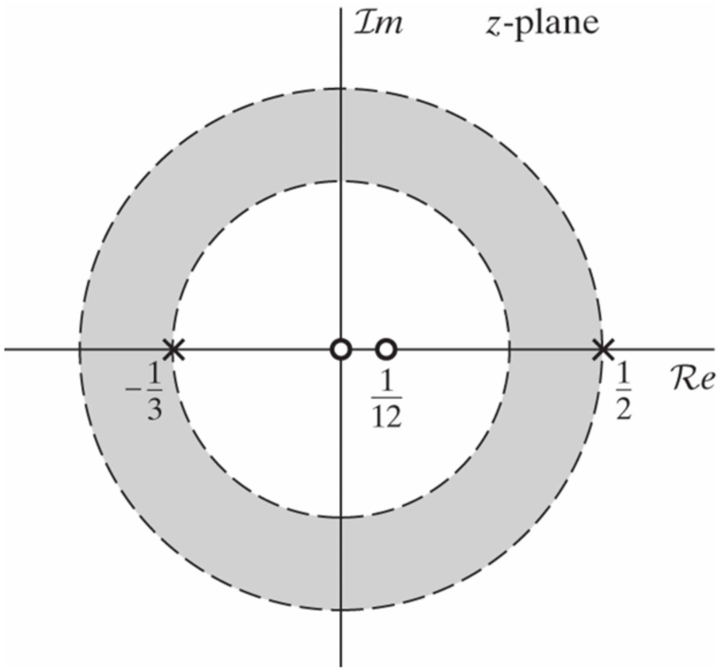
- Two-sided exponential sequence $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$



Finite-length sequence example I

- Recall z-transform $X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$

- No problem of convergence as long as $|x[n]z^{-n}|$ is finite

- May not have closed form expressions, may be unnecessary

$$x[n] = \delta[n] + \delta[n-5] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = 1 + z^{-5}$$

which is finite as long as $z \neq 0$

Finite-length sequence example 2

- Consider the sequence $x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

- z-transform becomes

$a < \infty$ and $z \neq 0$

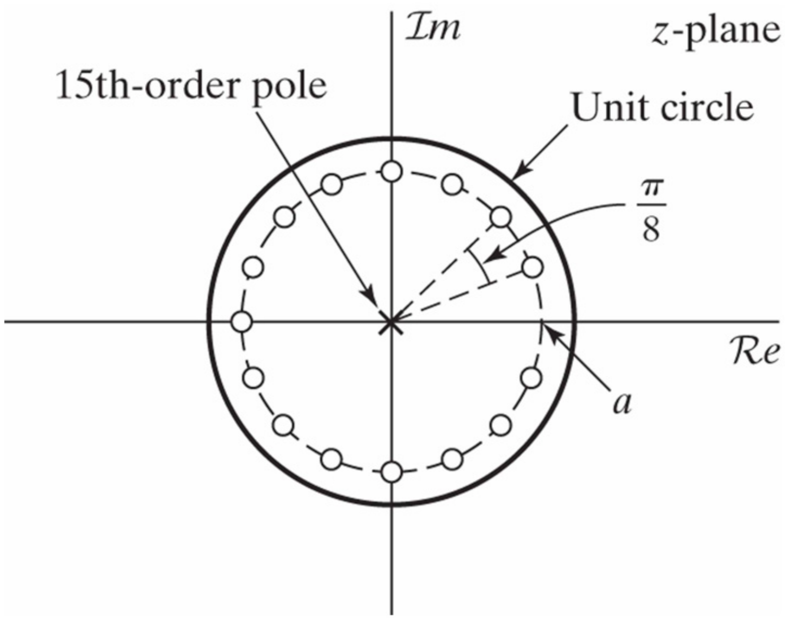
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

- N roots of numerator polynomial

$$z_k = ae^{j2\pi k/N}, \quad k = 0, 1, \ldots, N-1$$

Pole-zero cancellation with k=0



$N = 16, \; 0 < a < 1$

Common z-transform pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$